Review. A homogeneous linear DE with constant coefficients is of the form $p(D)y = 0$, where $p(D)$ is the characteristic polynomial polynomial. For each characteristic root r of multiplicity k, we get the k solutions $x^j e^{rx}$ for $j = 0, 1, \ldots, k-1$.

Example 68. Determine the general solution of $y^{(6)} = 3y^{(5)} - 4y'''$.

Solution. This DE is of the form $p(D)\,y = 0$ with $p(D) = D^6 - 3D^5 + 4D^3 = D^3(D-2)^2(D+1)$.
The characteristic roots are $2, 2, 0, 0, 0, -1$.
By Theorem 64, the general solution is $y(x) = (C_1 + C_2 x)e^{2x} + C_3 + C_4 x + C_5 x^2 + C_6 e^{-x}$.

Example 69. Consider the function $y(x) = 3x e^{-2x} + 7$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D)y = 0$, the characteristic roots must include $-2, -2, 0$.
The simplest choice for $p(D)$ thus is $p(D) = (D+2)^2 D = D^3 + 4D^2 + 4D$.
Accordingly, $y(x)$ is a solution of $y''' + 4y'' + 4y' = 0$.

Example 70. (homework) Consider the function $y(x) = 3x e^{-2x} + 7e^x$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D)y = 0$, the characteristic roots must include $-2, -2, 1$.
The simplest choice for $p(D)$ thus is $p(D) = (D+2)^2(D-1) = D^3 + 3D^2 - 4$.
Accordingly, $y(x)$ is a solution of $y''' + 3y'' - 4y = 0$.

Real form of complex solutions

Let's recall some basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.

- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at the examples below.

- The **real part** of $z = x + iy$ is x and we write $\mathrm{Re}(z) = x$.
  Likewise the **imaginary part** is $\mathrm{Im}(z) = y$.

  Observe that $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ as well as $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.

- **Euler's identity** (see Theorem 54) states that $e^{ix} = \cos(x) + i\sin(x)$.

  It follows that $\cos(x) = \mathrm{Re}(e^{ix}) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \mathrm{Im}(e^{ix}) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

Example 71. Determine the general solution of $y'' + y = 0$.

Solution. The characteristic polynomial is $D^2 + 1$ which has no roots over the reals.
Over the **complex numbers**, by definition, the roots are i and $-i$.
So the general solution is $y(x) = C_1 e^{ix} + C_2 e^{-ix}$.

Solution. On the other hand, we easily check that $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are two solutions.
Hence, the general solution can also be written as $y(x) = D_1 \cos(x) + D_2 \sin(x)$.

Important comment. That we have these two different representations is a consequence of Euler's identity (see Theorem 54)

$$e^{ix} = \cos(x) + i\sin(x).$$

Likewise, $e^{-ix} = \cos(x) - i\sin(x)$. (This follows from replacing x by $-x$ in Euler's identity.)

On the other hand, $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

[Recall that the first formula is an instance of $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ and the second of $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.]

Example 72. Determine the general solution of $y'' - 4y' + 13y = 0$ using only real numbers.

Solution. The characteristic polynomial $p(D) = D^2 - 4D + 13$ has roots $2 + 3i, 2 - 3i$.

[We can use the quadratic formula to find these roots as $\frac{4 \pm \sqrt{4^2 - 4 \cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$.]

Hence, the general solution in real form is $y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x)$.

Note. $e^{(2+3i)x} = e^{2x}e^{3ix} = e^{2x}(\cos(3x) + i\sin(3x))$