Review. Every homogeneous linear DE with constant coefficients can be written as $p(D)y=0$, where $D=\frac{\mathrm{d}}{\mathrm{d}x}$ and $p(D)$ is the characteristic polynomial polynomial.

Example 60. Determine the general solution of $y'''+7y''+14y'+8y=0$.

Solution. This DE is of the form $p(D)\,y=0$ with characteristic polynomial $p(D)=D^3+7D^2+14D+8$.

The characteristic polynomial factors as $p(D)=(D+1)(D+2)(D+4)$.

Hence, we found the solutions $y_1=e^{-x}$, $y_2=e^{-2x}$, $y_3=e^{-4x}$. Those are enough (independent!) solutions for a third-order DE. The general solution therefore is $y(x)=C_1\,e^{-x}+C_2\,e^{-2x}+C_3\,e^{-4x}$.

Example 61. Solve $y''=4y$ with initial conditions $y(0)=-1$, $y'(0)=10$.

Solution. This DE is of the form $p(D)\,y=0$ with characteristic polynomial $p(D)=D^2-4$.

The characteristic polynomial factors as $p(D)=(D-2)(D+2)$.

Hence, we found the solutions $y_1=e^{2x}$, $y_2=e^{-2x}$. Those are enough (independent!) solutions for a second-order DE. The general solution therefore is $y(x)=C_1\,e^{2x}+C_2\,e^{-2x}$.

Using $y'(x)=2C_1e^{2x}-2C_2e^{-2x}$, the initial conditions result in the equations $C_1+C_2=-1$, $2C_1-2C_2=10$.

Solving these we find $C_1=2$, $C_2=-3$.

Hence the unique solution to the IVP is $y(x)=2e^{2x}-3e^{-2x}$.

Comment (for those who have taken Linear Algebra). In matrix-vector notation, the two equations can be written and solved as

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\begin{bmatrix} -1 \\ 10 \end{bmatrix} \quad\rightsquigarrow\quad \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 10 \end{bmatrix}=-\frac{1}{4}\begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 10 \end{bmatrix}=\begin{bmatrix} 2 \\ -3 \end{bmatrix},$$

where we used the general formula for the inverse of a 2×2 matrix.

The system of equations $\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\begin{bmatrix} -1 \\ 10 \end{bmatrix}$ is an inhomogeneous system of linear equations.

The corresponding homogeneous system of linear equations is $\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Example 62. (extra) Determine the general solution of $y'''-y''-4y'+4y=0$.

Solution. This DE is of the form $p(D)\,y=0$ with characteristic polynomial $p(D)=D^3-D^2-4D+4$.

The characteristic polynomial factors as $p(D)=(D-1)(D-2)(D+2)$.

Hence, we found the solutions $y_1=e^{x}$, $y_2=e^{2x}$, $y_3=e^{-2x}$. Those are enough (independent!) solutions for a third-order DE. The general solution therefore is $y(x)=C_1\,e^{x}+C_2\,e^{2x}+C_3\,e^{-2x}$.

In this manner, we are able to solve any homogeneous linear DE of order n with constant coefficients provided that there are n different roots r (each giving rise to one solution e^{rx}).

One issue is that roots might be repeated. In that case, we are currently missing solutions. The following example suggests how to get our hands on the missing solutions.

Example 63. Determine the general solution of $y'''=0$.

Solution. We know from Calculus that the general solution is $y(x)=C_1+C_2\,x+C_3\,x^2$.

Solution. (looking ahead) The characteristic polynomial $p(D)=D^3$ has roots $0,0,0$. By Theorem 64 below, we have the solutions $y(x)=x^j\,e^{0x}=x^j$ for $j=0,1,2$, so that the general solution is $y(x)=C_1+C_2\,x+C_3\,x^2$.

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Theorem 64. Consider the homogeneous linear DE with constant coefficients $p(D)y=0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the DE are given by $x^j e^{rx}$ for $j=0,1,\ldots,k-1$.
- Combining these solutions for all roots, gives the general solution.

This is because the order of the DE equals the degree of $p(D)$, and a polynomial of degree n has (counting with multiplicity) exactly n (possibly **complex**) roots.

In the complex case. Likewise, if $r=a\pm bi$ are roots of the characteristic polynomial and if k is its multiplicity, then $2k$ (independent) solutions of the DE are given by $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$ for $j=0,1,\ldots,k-1$. This case will be discussed next time.

Proof. Let r be a root of the characteristic polynomial of multiplicity k. Then $p(D)=q(D)\,(D-r)^k$.

We need to find k solutions to the simpler DE $(D-r)^k y=0$.

It is natural to look for solutions of the form $y=c(x)e^{rx}$.

[This idea is called **variation of constants** since we know that this is a solution if $c(x)$ is a constant.]

Note that $(D-r)[c(x)e^{rx}]=(c'(x)e^{rx}+c(x)re^{rx})-rc(x)e^{rx}=c'(x)e^{rx}$.

Repeating, we get $(D-r)^2[c(x)e^{rx}]=(D-r)[c'(x)e^{rx}]=c''(x)e^{rx}$ and, eventually, $(D-r)^k[c(x)e^{rx}]=c^{(k)}(x)e^{rx}$.

In particular, $(D-r)^k y=0$ is solved by $y=c(x)e^{rx}$ if and only if $c^{(k)}(x)=0$.

The DE $c^{(k)}(x)=0$ is clearly solved by x^j for $j=0,1,\ldots,k-1$, and it follows that $x^j e^{rx}$ solves the original DE. ☐

Example 65. Determine the general solution of $y'''-3y''+3y'-y=0$.

Solution. The characteristic polynomial $p(D)=D^3-3D^2+3D-1=(D-1)^3$ has roots $1,1,1$.

By Theorem 64, the general solution is $y(x)=(C_1+C_2x+C_3x^2)e^x$.

Example 66. Determine the general solution of $y'''-3y'+2y=0$.

Solution. The characteristic polynomial $p(D)=D^3-3D+2=(D-1)^2(D+2)$ has roots $1,1,-2$.

By Theorem 64, the general solution is $y(x)=(C_1+C_2x)e^x+C_2e^{-2x}$.

Example 67. (homework) Solve the IVP $y'''=4y''-4y'$ with $y(0)=4$, $y'(0)=0$, $y''(0)=-4$.

Solution. The characteristic polynomial $p(D)=D^3-4D^2+4D=D(D-2)^2$ has roots $0,2,2$.

By Theorem 64, the general solution is $y(x)=C_1+(C_2+C_3x)e^{2x}$.

Using $y'(x)=(2C_2+C_3+2C_3x)e^{2x}$ and $y''(x)=4(C_2+C_3+C_3x)e^{2x}$, the initial conditions result in the equations $C_1+C_2=4$, $2C_2+C_3=0$, $4C_2+4C_3=-4$.

Solving these (start with the last two equations) we find $C_1=3$, $C_2=1$, $C_3=-2$.

Hence the unique solution to the IVP is $y(x)=3+(1-2x)e^{2x}$.