Linear DEs of higher order

The most general linear first-order DE is of the form $A(x)y' + B(x)y + C(x) = 0$. Any such DE can be rewritten in the form $y' + P(x)y = Q(x)$ by dividing by $A(x)$ and rearranging.

We have learned how to solve all of these using an integrating factor.

Likewise, any **linear DE** of order n can be written in the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = Q(x).$$

The corresponding **homogeneous linear DE** is the DE

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = 0,$$

and it plays an important role in solving the original linear DE.

Comment. The homogeneous equations $y' = F(y/x)$ that we encountered earlier are something rather different. It is a bit unfortunate that the word homogeneous is used for both.

Homogeneous linear DE have the important property that, if y_1 and y_2 are two solutions, then the linear combination $C_1\, y_1 + C_2\, y_2$ is a solution as well.

Example 56. Suppose that y_1 and y_2 solve $y'' + P_1(x)y' + P_0(x)y = 0$. Show that $7y_1 + 4y_2$ is another solution of the DE.

Solution. $(7y_1 + 4y_2)'' + P_1(x)(7y_1 + 4y_2)' + P_0(x)(7y_1 + 4y_2)$
$= 7\{y_1'' + P_1(x)y_1' + P_0(x)y_1\} + 4\{y_2'' + P_1(x)y_2' + P_0(x)y_2\} = 0 + 0$

In other words, $7y_1 + 4y_2$ is another solution of the DE.

Comment. Of course, there is nothing special about the coefficients 7 and 4. The same argument shows that any linear combination $C_1\, y_1 + C_2\, y_2$ is a solution as well.

Important comment. Make sure that you see that it is crucial that the DE is linear and that it is homogeneous! (What happens if the DE is linear but not homogeneous?)

The upshot is that this observation reduces the task of finding the general solution of a homogeneous linear DE to the task of finding n (sufficiently) different solutions.

(general solution of a homogeneous linear DE) For any homogeneous linear DE of order n, there are n solutions $y_1, y_2, \ldots, y_n$ such that the general solution is

$$y(x) = C_1 y_1(x) + \ldots + C_n\, y_n(x).$$

Comment. As we observed in the first-order case, if I is an interval on which all the $P_j(x)$ as well as $P(x)$ are continuous, then for any $a \in I$ the IVP with $y(a) = b_0$, $y'(a) = b_1$, ..., $y^{(n-1)}(a) = b_{n-1}$ always has a unique solution (which is defined on all of I).

Example 57. (extra) The DE $x^2 y'' + 2x y' - 6y = 0$ has solutions $y_1 = x^2$, $y_2 = x^{-3}$. Solve the IVP with $y(2) = 10$, $y'(2) = 15$.

Solution. Note that this is a homogeneous linear DE of order 2.

Hence, given the two solutions, we conclude that the general solution is $y(x) = Ax^2 + Bx^{-3}$.

Using $y'(x) = 2Ax - 3Bx^{-4}$, the two initial conditions allow us to solve for A and B:

Solving $y(2) = 4A + B/8 = 10$ and $y'(2) = 4A - 3/16B = 15$ for A and B results in $A = 3$, $B = -16$.

So the unique solution to the IVP is $y(x) = 3x^2 - 16/x^3$.

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Homogeneous linear DEs with constant coefficients

Let us start with another example like Examples 6 and 51. This time we also approach this computation using an operator approach that explains further what is going on (and that will be particularly useful when we discuss inhomogeneous equations).

An **operator** takes a function as input and returns a function as output. That is exactly what the derivative does.

In the sequel, we write $D = \frac{\mathrm{d}}{\mathrm{d}x}$ for the derivative operator.

For instance. We write $y' = \frac{\mathrm{d}}{\mathrm{d}x} y = D y$ as well as $y'' = \frac{\mathrm{d}^2}{\mathrm{d}x^2} y = D^2 y$.

Example 58. Find the general solution to $y'' - y' - 2y = 0$.

Solution. (our earlier approach) Let us look for solutions of the form e^{rx}.
Plugging e^{rx} into the DE, we get $r^2 e^{rx} - r e^{rx} - 2e^{rx} = 0$.
Equivalently, $r^2 - r - 2 = 0$. This is the characteristic equation. Its solutions are $r = 2, -1$.
This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.
Since this a homogeneous linear DE, the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Solution. (operator approach) $y'' - y' - 2y = 0$ is equivalent to $(D^2 - D - 2) y = 0$.
Note that $D^2 - D - 2 = (D-2)(D+1)$ is the **characteristic polynomial**.
Observe that we get solutions to $(D-2)(D+1) y = 0$ from $(D-2) y = 0$ and $(D+1) y = 0$.
$(D-2) y = 0$ is solved by $y_1 = e^{2x}$, and $(D+1) y = 0$ is solved by $y_2 = e^{-x}$; as in the previous solution.
Again, we conclude that the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Set $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Every **homogeneous linear DE with constant coefficients** can be written as $p(D) y = 0$, where $p(D)$ is a polynomial in D, called the **characteristic polynomial**.

For instance. $y'' - y' - 2y = 0$ is equivalent to $L y = 0$ with $L = D^2 - D - 2$.

Example 59. Solve $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

Solution. From the previous example, we know that the general solution is $y(x) = C_1 e^{2x} + C_2 e^{-x}$.
Using $y'(x) = 2C_1 e^{2x} - C_2 e^{-x}$, the initial conditions result in the two equations $C_1 + C_2 = 4$, $2C_1 - C_2 = 5$.
Solving these we find $C_1 = 3$, $C_2 = 1$.
Hence the unique solution to the IVP is $y(x) = 3e^{2x} + e^{-x}$.

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