Let us have another look at Example 6. Note that the DE is a second-order linear differential equation with constant coefficients. Our upcoming goal will be to solve all such equations.

Example 51. Find the general solution to $y'' = y' + 6y$.

Solution. We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^2 e^{rx} = r e^{rx} + 6e^{rx}$ which simplifies to $r^2 - r - 6 = 0$.

This is called the **characteristic equation**. Its solutions are $r = -2, 3$ (the **characteristic roots**).

This means we found the two solutions $y_1 = e^{-2x}$, $y_2 = e^{3x}$.

The general solution to the DE is $C_1 e^{-2x} + C_2 e^{3x}$.

Comment. In the final step, we used an important principle that is true for linear (!) homogeneous DEs. Namely, if we have solutions $y_1, y_2, \ldots$ then any linear combination $C_1 y_1 + C_2 y_2 + \ldots$ is a solution as well. We will discuss this soon but, for now, check that $C_1 e^{-2x} + C_2 e^{3x}$ is indeed a solution by plugging it into the DE.

Spotlight on the exponential function

Example 52. Solve $y' = ky$ where k is a constant.

Solution. (experience) At this point, we can probably see that $y(x) = e^{kx}$ is a solution.

In fact, the general solution is $y(x) = Ce^{kx}$.

That there cannot be any further solutions follows from the existence and uniqueness theorem (see next example).

Solution. (separation of variables) Alternatively, we can solve the DE using separation of variables.

Express the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$, then write it as $\frac{1}{y}\mathrm{d}y = k\mathrm{d}x$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = kx + D$, hence $|y| = e^{kx+D}$.

Since the RHS is never zero, $y = \pm e^{kx+D} = Ce^{kx}$ (with $C = \pm e^D$). Finally, note that $C = 0$ corresponds to the singular solution $y = 0$ that we lost. In summary, the general solution is Ce^{kx}.

Example 53. Consider the IVP $y' = ky$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = ky$. We compute that $\frac{\partial}{\partial y} f(x, y) = k$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Comment. As a consequence, there can be no other solutions to the DE $y' = ky$ than the ones of the form $y(x) = Ce^{kx}$. Why?! [Assume that $y(x)$ satisfies $y' = ky$ and let (a, b) any value on the graph of y. Then $y(x)$ solves the IVP $y' = ky$, $y(a) = b$; but so does Ce^{kx} with $C = b/e^{ka}$. The uniqueness implies that $y(x) = Ce^{kx}$.]

In particular, we have the following characterization of the exponential function:

e^x is the unique solution to the IVP $y' = y$, $y(0) = 1$.

Comment. Note that, for instance, $\frac{\mathrm{d}}{\mathrm{d}x} 2^x = \ln(2)\, 2^x$. (This follows from $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$.)

Since $\ln = \log_e$, this means that we cannot avoid the natural base $e \approx 2.718$ even if we try to use another base.

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Excursion: Euler's identity

Theorem 54. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Example 55. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?
Or, use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = \ldots$

Realize that the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$.
These are precisely the points on the unit circle!

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), we can write

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

In the final step, we used Euler's identity.

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