Review. The **logistic equation** is $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$.

Here, k is the growth rate and M is the carrying capacity.

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Example 47. In a city with a fixed population N, the time rate of change of the number P of people who have heard a certain rumor is proportional to the product of P and $N - P$. Suppose initially 10% have heard the rumor and after a week this number has grown to 20%. What percentage will this number reach after one more week?

Solution. $\frac{dP}{dt} = \gamma P(N - P)$. $P(0) = 0.1N$ and $P(1) = 0.2N$. We need $P(2)$.

Note that this is a logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$ with $k = \gamma N$ and carrying capacity N.

It therefore has the general solution $P(t) = \frac{N}{1 + Ce^{-kt}}$.

Using $P(0) = \frac{N}{1 + C} = 0.1N$, we find that $C = 9$.

Using $P(1) = \frac{N}{1 + 9e^{-k}} = 0.2N$, we further find that $e^{-k} = \frac{4}{9}$.

We could solve for k but note that it is more pleasing to use $e^{-kt} = (e^{-k})^t = \left(\frac{4}{9}\right)^t$ in our formula for $P(t)$.

We conclude that $P(t) = \frac{N}{1 + 9\left(\frac{4}{9}\right)^t}$.

In particular, $P(2) = \frac{N}{1 + 9 \cdot \frac{16}{81}} = \frac{9}{25}N$ which is 36%.

Mixing problems

Example 48. A tank contains 20gal of pure water. It is filled with brine (containing 5lb/gal salt) at a rate of 3gal/min. At the same time, well-mixed solution flows out at a rate of 2gal/min. How much salt is in the tank after t minutes?

Solution. Let $x(t)$ denote the amount of salt (in lb) in the tank after time t (in min).

At time t, the concentration of salt (in lb/gal) in the tank is $\frac{x(t)}{V(t)}$ where $V(t) = 20 + (3 - 2)t = 20 + t$ is the volume (in gal) in the tank.

In the time interval $[t, t + \Delta t]$: $\Delta x \approx 3 \cdot 5 \cdot \Delta t - 2 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$.

Hence, $x(t)$ solves the IVP $\frac{dx}{dt} = 15 - 2 \cdot \frac{x}{20 + t}$ with $x(0) = 0$. Since this is a linear DE, we can solve it as follows:

- We write it in the form $\frac{dx}{dt} + \frac{2}{20 + t}x = 15$.
- The integrating factor is $f(t) = \exp\left(\int \frac{2}{20 + t} dt\right) = (20 + t)^2$.
- Multiply the (rewritten) DE by $f(t) = (20 + t)^2$ to get $\underbrace{(20 + t)^2 \frac{dx}{dt} + 2(20 + t)x}_{= \frac{d}{dt}[(20+t)^2 x]} = 15(20 + t)^2$.
- Integrate both sides to get $(20 + t)^2 x = 15 \int (20 + t)^2 dt = 5(20 + t)^3 + C$.

Hence the general solution to the DE is $x(t) = 5(20 + t) + \frac{C}{(20 + t)^2}$. Using $x(0) = 0$, we find $C = -5 \cdot 20^3$.

We conclude that, after t minutes, the tank contains $x(t) = 5(20 + t) - \frac{5 \cdot 20^3}{(20 + t)^2}$ pounds of salt.

Comment. As a consequence, $x(t) \approx 5(20 + t) = 5V(t)$ for large t. Why does that make perfect sense?!

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