The logistic model of population growth

If the population is constrained by resources, then $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ is not a good model. A model to take that into account is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$. This is the **logistic equation**.

M is called the carrying capacity:

- Note that if $P \ll M$ then $1 - \frac{P}{M} \approx 1$ and we are back to the simpler exponential model. This means that the population P will grow (nearly) exponentially if P is much less than the carrying capacity M.
- On the other hand, if $P > M$ then $1 - \frac{P}{M} < 0$ so that (assuming $k > 0$) $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$, which means that the population P is shrinking if it exceeds the carrying capacity M.

Comment. If $P(t)$ is the size of a population, then P'/P can be interpreted as its *per capita growth rate*.
Note that in the exponential model we have that $P'/P = k$ is constant.
On the other hand, in the logistic model we have that $P'/P = k(1 - P/M)$ is a linear function.

Example 43. Solve the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Solution. This is a separable DE: $\frac{1}{P(1-\frac{P}{M})}\mathrm{d}P = k\,\mathrm{d}t$.

To integrate the left-hand side, we use partial fractions: $\frac{1}{P(1-\frac{P}{M})} = \frac{1}{P} + \frac{1/M}{1-\frac{P}{M}} = \frac{1}{P} - \frac{1}{P-M}$.

After integrating, we obtain $\ln|P| - \ln|P-M| = kt + A$.

Equivalently, $\ln\left|\frac{P}{P-M}\right| = kt + A$ so that $\frac{P}{P-M} = \pm e^{kt+A} = Be^{kt}$ where $B = \pm e^A$.

Solving for P, we conclude that the general solution is

$$P(t) = \frac{BMe^{kt}}{Be^{kt} - 1} = \frac{M}{1 + Ce^{-kt}},$$

where replaced the free parameter B with $C = -1/B$.

Initial population. Note that the initial population is $P(0) = \frac{M}{1+C}$. Equivalently, $C = \frac{M}{P(0)} - 1$ which expresses the free parameter C in terms of the initial population.

Comment. Note that $B = \pm e^A$ can be any real number except 0. However, we can easily check that $B = 0$ also gives us a solution to the DE (namely, the trivial solution $P = 0$). This solution was "lost" when we divided by P to separate variables.

Exercise. Note that the logistic equation is a Bernoulli equation. As an alternative to separation of variables, we can therefore solve it by transforming it to a linear DE.

Review of partial fractions. Recall that partial fractions tells us that fractions like $\frac{p(x)}{(x-r_1)(x-r_2)\cdots}$ (with the numerator of smaller degree than the denominator; and with the r_j distinct) can be written as a sum of terms of the form $\frac{A_j}{x-r_j}$ for suitable constants A_j.

In our case, this tells us that $\frac{1}{P(1-P/M)} = \frac{A}{P} + \frac{B}{1-P/M}$ for certain constants A and B.

Multiply both sides by P and set $P = 0$ to find $A = 1$.

Multiply both sides by $1 - P/M$ and set $P = M$ to find $B = 1/M$. This is what we used above.

The **logistic equation** with growth rate k and carrying capacity M is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right).$$

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Armin Straub
straub@southalabama.edu

Example 44. Let $P(t)$ describe the size of a population at time t. Under the logistic model of population growth, what is $\lim_{t\to\infty} P(t)$?

Solution.

- If $k>0$, then $e^{-kt}\to 0$ and it follows from $P(t)=\frac{M}{1+Ce^{-kt}}$ that $\lim_{t\to\infty} P(t)=M$.
  In other words, the population will approach the carrying capacity in the long run.
- If $k=0$, then we simply have $P(t)=\frac{M}{1+C}$. In other words, the population remains constant.
  This is a corner case because the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t}=0$.
- If $k<0$, then $e^{-kt}\to\infty$ and it follows that $\lim_{t\to\infty} P(t)=0$.
  In other words, the population will approach extinction in the long run.

Example 45. (homework) A rising population is modeled by the equation $\frac{\mathrm{d}P}{\mathrm{d}t}=400P-2P^2$.

(a) When the population size stabilizes in the long term, how big will the population be?

(b) Under which condition will the population size shrink?

(c) What is the population size when it is growing the fastest?

(d) If $P(0)=10$, what is $P(t)$?

Solution.

(a) Once the population reaches a stable level in the long term, we have $\frac{\mathrm{d}P}{\mathrm{d}t}=0$ (no change in population size).
Hence, $0=400P-2P^2=2P(200-P)$ which implies that $P=0$ or $P=200$. Since the population is rising, it will approach 200 in the long term.
Alternatively. Our DE matches the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{M}\right)$ with $k=400$ and $M=200$.

(b) The population size will shrink if $\frac{\mathrm{d}P}{\mathrm{d}t}<0$.
The DE tells us that is the case if and only if $400P-2P^2<0$ or, equivalently, if $P>\frac{400}{2}=200$.
Comment. In the logistic model, the population shrinks if it exceeds the carrying capacity.

(c) This is asking when $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the population growth) is maximal.
The DE is telling us that this growth is $f(P)=400P-2P^2$. This a parabola that opens to the bottom. From Calculus, we know that it has a global maximum when $f'(P)=0$.
$f'(P)=400-4P=0$ leads to $P=100$.
Thus, the population is growing the fastest when its size is 100.
Comment. In the logistic model, the population is growing fastest when it is half the carrying capacity.

(d) We know that the general solution of the logistic equation is $P(t)=\frac{M}{1+Ce^{-kt}}=\frac{200}{1+Ce^{-400t}}$.
Using $P(0)=10$, we find that $C=\frac{200}{10}-1=19$.
Thus $P(t)=\frac{200}{1+19e^{-400t}}$.

Example 46. A scientist is claiming that a certain population $P(t)$ follows the logistic model of population growth perfectly. How many data points do you need to begin to verify that claim?

Solution. The general solution $P(t)=\frac{M}{1+Ce^{-kt}}$ to the logistic equation has 3 parameters.

Hence, we need 3 data points just to solve for their values.

Once we have 4 or more data points, we are able to test whether $P(t)$ conforms to the logistic model.

Important comment. Complicated models tend to have many degrees of freedom, which makes it easier to fit them to real world data (even if the model is not actually particularly appropriate). We therefore need to be cognizant about how much evidence is needed to decide that a given model is appropriate for the data.

Further population models

Let $P(t)$ be the size of the population that we wish to model at time t.

Denote with $\beta(t)$ and $\delta(t)$ the birth and death rate at time t, measured in number of births or deaths per unit of population per unit of time.

In the time interval $[t, t+\Delta t]$, we have that

$$\Delta P \approx \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t.$$

Comment. The reason that this is not an exact equation is that the rates $\beta(t)$ and $\delta(t)$ are allowed to change with t. In the above, we used these rates at time t for all times in $[t, t+\Delta t]$. This is a good approximation if Δt is small.

Divide both sides by Δt and let $\Delta t \to 0$ to obtain the general differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P.$$

Given certain scenarios, we now make corresponding reasonable choices for $\beta(t)$ and $\delta(t)$.

- **(basic)** If the rates $\beta(t)$ and $\delta(t)$ are constant over time, the DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta - \delta)P$.
  This is the exponential model of population growth.

- **(limited supply)** If supply is limited, the birth rate will decrease as P increases. The simplest such relationship would be a linear dependence, which would take the form $\beta(t) = \beta_0 - \beta_1 P$.

  On the other hand, we still assume that $\delta(t)$ is constant. (However, depending on circumstances, it could also be reasonable to assume that $\delta(t)$ increases as P increases.)

  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta_0 - \beta_1 P - \delta)P$.

  This is the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(1 - P/M)$ with $k = \beta_0 - \delta$ and $\frac{k}{M} = \beta_1$.

- **(rare isolated species)** If the population consists of rare and isolated specimen which rely on chance encounters to reproduce, then it is reasonable to assume that the birth rate $\beta(t)$ is proportional to $P(t)$ (larger $P(t)$ means more possibilities for chance encounters). Once more, we assume that $\delta(t)$ constant.

  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (kP - \delta)P$.

  This is, again, the logistic equation.

- **(rare isolated species with very long life)** As before, for a rare isolated population, it is reasonable to assume that $\beta(t)$ is proportional to $P(t)$. If, in addition, our specimen have very long life, then we would assume that $\delta(t) = 0$.

  The corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2$. Solutions are $P(t) = \frac{1}{C - kt}$ where $P(0) = 1/C$. (Do it!)

  Comment. Note that $P(t) \to \infty$ as $t \to C/k$. This explosion (which implies population growth beyond exponential growth) emphasizes that we can only use the DE while our initial assumptions are satisfied. Here, the DE is no longer appropriate when our species is no longer rare because $P(t)$ is too large.

- **(spread of contagious incurable virus)** Let $P(t)$ count the number of infected population units among a (constant) total of N. Since the virus is incurable, we have $\delta(t) = 0$. On the other hand, it is reasonable to assume that $\beta(t)$ is proportional to $N - P$ (the number of people that can still be infected).

  The resulting DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(N - P)$. Once again, this is the logistic equation.

- **(harvesting)** Suppose that h population units are harvested each unit of time.

  Then the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P - h$.

  For instance. $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - h$ has the solution $P(t) = Ce^{kt} + h/k$. In that case, we get exponential growth if $C > 0$. Note that $P(0) = C + h/k$. In terms of the initial population $P(0)$, we therefore get exponential growth if $P(0) > h/k$.

Armin Straub
straub@southalabama.edu