Example 35. Consider $\frac{dy}{dx} = F(x)y + G(x)y^n$. This is called a **Bernoulli equation**. Substitute $u = y^{1-n}$ and show that the resulting linear DE.

Solution. If $u = y^{1-n}$ then $y = u^{1/(1-n)}$ and, thus, $\frac{dy}{dx} = \frac{1}{1-n}u^{n/(1-n)}\frac{du}{dx}$. $[\frac{1}{1-n} - 1 = \frac{n}{1-n}]$

The new DE is $\frac{1}{1-n}u^{n/(1-n)}\frac{du}{dx} = F(x)u^{1/(1-n)} + G(x)u^{n/(1-n)}$.

Dividing both sides by $u^{n/(1-n)}$, the DE simplifies to $\frac{1}{1-n}\frac{du}{dx} = F(x)u + G(x)$.

Comment. The original DE has the trivial solution $y = 0$. Do you see where we might lose that solution?

Example 36. (homework) Solve the IVP $\frac{dy}{dx} = 2y - 3xy^5$, $y(0) = 1$.

Solution. This is an example of a Bernoulli equation (with $n = 5$). We therefore substitute $u = y^{1-n} = y^{-4}$.
Accordingly, $y = u^{-1/4}$ and, thus, $\frac{dy}{dx} = -\frac{1}{4}u^{-5/4}\frac{du}{dx}$.
The new DE is $-\frac{1}{4}u^{-5/4}\frac{du}{dx} = 2u^{-1/4} - 3xu^{-5/4}$, which simplifies to $\frac{du}{dx} = -8u + 12x$.
This is a linear first-order DE, which we solve according to our recipe:

(a) Rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 8$ and $Q(x) = 12x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^{8x}$.

(c) Multiply the (rewritten) DE by $f(x) = e^{8x}$ to get

$$\underbrace{e^{8x}\frac{du}{dx} + 8e^{8x}u}_{= \frac{d}{dx}[e^{8x}u]} = 12xe^{8x}.$$

(d) Integrate both sides to get:

$$e^{8x}u = 12\int xe^{8x}dx = 12\left(\frac{1}{8}xe^{8x} - \frac{1}{8^2}e^{8x}\right) + C = \frac{3}{2}xe^{8x} - \frac{3}{16}e^{8x} + C$$

Here we used that $\int xe^{ax}dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$. (Integration by parts!)

The general solution of the DE for u therefore is $u = \frac{3}{2}x - \frac{3}{16} + Ce^{-8x}$.
Correspondingly, the general solution of the initial DE is $y = u^{-1/4} = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + Ce^{-8x}}$.
Using $y(0) = 1$, we find $1 = 1/\sqrt[4]{C - \frac{3}{16}}$ from which we obtain $C = 1 + \frac{3}{16} = \frac{19}{16}$.
The unique solution to the IVP therefore is $y = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + \frac{19}{16}e^{-8x}}$.

Armin Straub
straub@southalabama.edu

Example 37. Solve $(x-y)\frac{dy}{dx} = x+y$.

Solution. Divide the DE by x to get $(1-\frac{y}{x})\frac{dy}{dx} = 1+\frac{y}{x}$. This is a homogeneous equation!

We therefore substitute $u = \frac{y}{x}$. Then $y = ux$ and $\frac{dy}{dx} = x\frac{du}{dx} + u$.

The resulting DE is $(x-ux)\left(x\frac{du}{dx}+u\right) = x + ux$, which simplifies to $x(1-u)\frac{du}{dx} = 1+u^2$.

This DE is separable: $\frac{1-u}{1+u^2}du = \frac{1}{x}dx$

Integrating both sides, we find $\arctan(u) - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$.

Setting $u = y/x$, we get the (general) implicit solution $\arctan(y/x) - \frac{1}{2}\ln(1+(y/x)^2) = \ln|x| + C$.

Comment. We used $\int \frac{1}{1+u^2}du = \arctan(u) + C$ and $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) + C$ when integrating.

See Example 31 where we reviewed these integrals.

Solving simple 2nd order DEs

We have the following two useful substitutions for certain simple DEs of order 2:

- $F(y'', y', x) = 0$ (2nd order with "y missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx}$. We get the first-order DE $F\left(\frac{du}{dx}, u, x\right) = 0$.

- $F(y'', y', y) = 0$ (2nd order with "x missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$. We get the first-order DE $F\left(u\frac{du}{dy}, u, y\right) = 0$.

Example 38. Solve $y'' = x - y'$.

Solution. We substitute $u = y'$, which results in the first-order DE $u' = x - u$.

This DE is linear and, using our recipe (see below for the details), we can solve it to find $u = x - 1 + Ce^{-x}$.

Since $y' = u$, we conclude that the general solution is $y = \int (x - 1 + Ce^{-x})dx = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

Important comment. This is a DE of order 2. Hence, as expected, the general solution has two free parameter.

Solving the linear DE. To solve $u' = x - u$, we

(a) rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 1$ and $Q(x) = x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^x$.

(c) Multiply the (rewritten) DE by $f(x) = e^x$ to get $\underbrace{e^x\frac{du}{dx} + e^x u}_{= \frac{d}{dx}[e^x u]} = xe^x$.

(d) Integrate both sides to get (using integration by parts): $e^x u = \int xe^x dx = xe^x - e^x + C$

Hence, the general solution of the DE for u is $u = x - 1 + Ce^{-x}$, which is what we used above.

Armin Straub
straub@southalabama.edu

Example 39. **(extra)** Find the general solution to $y'' = 2yy'$.

Solution. We substitute $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$.

Therefore, our DE turns into $u\frac{du}{dy} = 2yu$.

Dividing by u, we get $\frac{du}{dy} = 2y$. [Note that we lose the solution $u = 0$, which gives the singular solution $y = C$.]

Hence, $u = y^2 + C$. It remains to solve $y' = y^2 + C$. This is a separable DE.

$\frac{1}{C + y^2}dy = dx$. Let us restrict to $C = D^2 \geqslant 0$ here. (This means we will only find "half" of the solutions.)

$\int \frac{1}{D^2 + y^2}dy = \frac{1}{D^2}\int \frac{1}{1 + (y/D)^2}dy = \frac{1}{D}\arctan(y/D) = x + A$.

Solving for y, we find $y = D\tan(Dx + AD) = D\tan(Dx + B)$. $[B = AD]$

Applications of DEs & Modeling

The exponential model of population growth

If $P(t)$ is the size of a population (eg. of bacteria) at time t, then the rate of change $\frac{dP}{dt}$ might, from biological considerations, be (nearly) proportional to $P(t)$.

Comment. "Population" might sound more specific than it is. It could also refer to rather different populations such as amounts of money (finance) or amounts of radioactive material (physics).

For instance, thinking about an amount $P(t)$ of money in a bank account at time t, we would also expect $\frac{dP}{dt}$ (the money per time that we gain from receiving interest) to be proportional to $P(t)$.

The corresponding **mathematical model** is described by the DE $\frac{dP}{dt} = kP$ where k is the constant of proportionality.

Example 40. Determine all solutions to the DE $\frac{dP}{dt} = kP$.

Solution. We easily guess and then verify that $P(t) = Ce^{kt}$ is a solution. (Alternatively, we can find this solution via separation of variables or because this is a linear DE. Do it both ways!)

Moreover, it follows from the existence and uniqueness theorem that there cannot be further solutions. (Alternatively, we can conclude this from our solving process (separation of variables or our approach to linear DEs only lose solutions when we divide by zero and we can consider those cases separately)).

Mathematics therefore tells us that the (only) solutions to this DE are given by $P(t) = Ce^{kt}$ where C is some constant.

Hence, populations satisfying the assumption from biology necessarily exhibit exponential growth.

Example 41. Let $P(t)$ describe the size of a population at time t. Suppose $P(0) = 100$ and $P(1) = 300$. Under the exponential model of population growth, find $P(t)$.

Solution. $P(t)$ solves the DE $\frac{dP}{dt} = kP$ and therefore is of the form $P(t) = Ce^{kt}$.

We now use the two data points to determine both C and k.

$Ce^{k\cdot 0} = C = 100$ and $Ce^k = 100e^k = 300$. Hence $k = \ln(3)$ and $P(t) = 100e^{\ln(3)t} = 100 \cdot 3^t$.

Main problem of modeling: a model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.

Armin Straub
straub@southalabama.edu