Notes for Lecture 5

Tue, 8/30/2022

Review. Existence and uniqueness theorem (Theorem 19) for an IVP $y' = f(x,y)$, $y(a) = b$:
If $f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are continuous around (a,b) then, locally, the IVP has a unique solution.

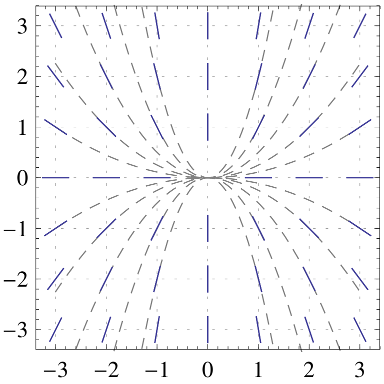
Example 21. Consider, again, the IVP $xy' = 2y$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x,y)$ with $f(x,y) = 2y/x$.

We compute that $\frac{\partial}{\partial y} f(x,y) = 2/x$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are continuous for all (x,y) with $x \neq 0$.

Hence, if $a \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.



What happens in the case $a = 0$?

Solution. In Example 15, we found that the DE $xy' = 2y$ is solved by $y(x) = Cx^2$.

This means that the IVP with $y(0) = 0$ has infinitely many solutions.

On the other hand, the IVP with $y(0) = b$ where $b \neq 0$ has no solutions. (This follows from the fact that there are no solutions to the DE besides $y(x) = Cx^2$. Can you see this by looking at the slope field?)

Example 22. Consider the IVP $y' = ky^2$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x,y)$ with $f(x,y) = ky^2$. We compute that $\frac{\partial}{\partial y} f(x,y) = 2ky$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are continuous for all (x,y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 23. Solve $y' = ky^2$.

Solution. Separate variables to get $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = k$.

Integrating $\int \frac{1}{y^2}\mathrm{d}y = \int k\,\mathrm{d}x$, we find $-\frac{1}{y} = kx + C$.

We solve for y to get $y = -\frac{1}{C + kx} = \frac{1}{D - kx}$ (with $D = -C$). That is the solution we verified earlier!

Comment. Note that we did not find the solution $y = 0$ (it was "lost" when we divided by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). However, note that we can obtain it from the general solution by letting $D \to \infty$.

Caution. We have to be careful about transforming our DE when using separation of variables: Just as the division by y^2 made us lose a solution, other transformations can add extra solutions which do not solve the original DE.

Here is a silly example (silly, because the transformation serves no purpose here) which still illustrates the point. The DE $(y-1)y' = (y-1)ky^2$ has the same solutions as $y' = ky^2$ plus the additional solution $y = 1$ (which does not solve $y' = ky^2$).

Example 24. (extra) Solve the IVP $y' = y^2$, $y(0) = 1$.

Solution. From the previous example with $k = 1$, we know that $y(x) = \frac{1}{D - x}$.

Using $y(0) = 1$, we find that $D = 1$ so that the unique solution to the IVP is $y(x) = \frac{1}{1 - x}$.

Comment. Note that we already concluded the uniqueness from the existence and uniqueness theorem.

On the other hand, note that $y(x) = \frac{1}{1 - x}$ is only valid on $(-\infty, 1)$ and that it cannot be continuously extended past $x = 1$; it is only a local solution.

Example 25. (homework) Consider the IVP $(x-y^2)y'=3x$, $y(4)=b$. For which choices of b does the existence and uniqueness theorem guarantee a unique (local) solution?

Solution. The IVP is $y'=f(x,y)$ with $f(x,y)=3x/(x-y^2)$. We compute that $\frac{\partial}{\partial y}f(x,y)=6xy/(x-y^2)^2$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous for all (x,y) with $x-y^2\neq 0$.

Note that $4-b^2\neq 0$ is equivalent to $b\neq\pm 2$.

Hence, if $b\neq\pm 2$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Linear first-order DEs

A **linear differential equation** is one where the function y and its derivatives only show up linearly (i.e. there is nothing like y^2, $1/y$ or $\sin(y)$).

As such, the most general linear first-order DE is of the form

$$A(x)y'+B(x)y+C(x)=0.$$

Comment. Note that any such DE can be rewritten in the form $y'+P(x)y=Q(x)$ by dividing by $A(x)$ and rearranging.

Example 26. (extra) Solve $\frac{\mathrm{d}y}{\mathrm{d}x}=2xy^2$.

Solution. (separation of variables) $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=2x$, $-\frac{1}{y}=x^2+C$.

Hence the general solution is $y=\frac{1}{D-x^2}$. [There also is the singular solution $y=0$.]

Solution. (in other words) Note that $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=2x$ can be written as $\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{y}\right]=\frac{\mathrm{d}}{\mathrm{d}x}[x^2]$.

From there it follows that $-\frac{1}{y}=x^2+C$, as above.

We now use the idea of writing both sides as a derivative to also solve linear DEs that are not separable.

The multiplication by $\frac{1}{y^2}$ will be replaced by multiplication with a so-called **integrating factor**.

Example 27. Solve $y'=x-y$.

Comment. Note that we cannot use separation of variables this time.

Solution. Rewrite the DE as $y'+y=x$.

Next, multiply both sides with e^x (we will see in a little bit how to find this "integrating factor") to get

$$\underbrace{e^xy'+e^xy}_{=\frac{\mathrm{d}}{\mathrm{d}x}[e^xy]}=xe^x.$$

The "magic" part is that we are able to realize the new left-hand side as a derivative!

Next, we will integrate both sides and then solve for y. (Try it yourself!) To be continued…

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