Example 15. Consider the DE $xy' = 2y$.

Sketch its slope field.

  Challenge. Try to guess solutions $y(x)$ from the slope field.

  Solution. For instance, to find the slope at the point $(3,1)$, we plug $x = 3$, $y = 1$ into the DE to get $3y' = 2$. Hence, the slope is $y' = 2/3$.

  The resulting slope field is sketched on the right.

  Solution of the challenge. Trace out the solution through $(1,1)$ (and then some other points). Their shape looks like a parabola, so that we might guess that $y(x) = Cx^2$ solves the DE. Check that this is indeed the case by plugging into the DE!
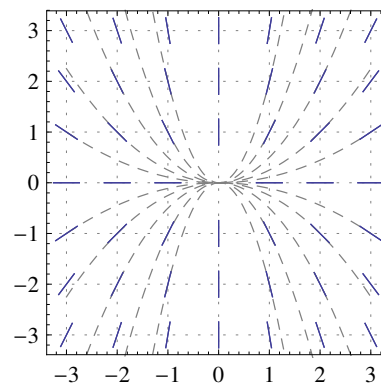
Example 16. Solve the IVP $xy' = 2y$, $y(1) = 2$.

  Solution. Rewrite the DE as $\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$.

  Then multiply both sides with dx and integrate both of them to get $\int \frac{1}{y}dy = \int \frac{2}{x}dx$.

  Hence, $\ln|y| = 2\ln|x| + C$.

  The initial condition $y(1) = 2$ tells us that, at least locally, $x > 0$ and $y > 0$. Thus $\ln(y) = 2\ln(x) + C$.

  Moreover, plugging in $x = 1$ and $y = 2$, we find $C = \ln(2)$.

  Solving $\ln(y) = 2\ln(x) + \ln(2)$ for y, we find $y = e^{2\ln(x) + \ln(2)} = 2x^2$.

  Comment. When solving a DE or IVP, we can generally only expect to find a **local solution**, meaning that our solution might only be valid in a small interval around the initial condition (here, we can only expect $y(x)$ to be a solution for all x in an interval around 1; especially since we assumed $x > 0$ in our solution). However, we can check (do it!) that the solution $y = 2x^2$ is actually a **global solution** (meaning that it is a solution for all x, not just locally around 1).

Example 17. Solve the IVP $xy' = 2y$, $y(1) = -1$.

  Solution. Again, we rewrite the DE as $\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$, multiply both sides with dx, and integrate to get $\int \frac{1}{y}dy = \int \frac{2}{x}dx$.

  Hence, $\ln|y| = 2\ln|x| + C$. The initial condition $y(1) = -1$ tells us that, at least locally, $x > 0$ and $y < 0$ (note that this means $|y| = -y$). Thus $\ln(-y) = 2\ln(x) + C$.

  Moreover, plugging in $x = 1$ and $y = -1$, we find $C = 0$.

  Solving $\ln(-y) = 2\ln(x)$ for y, we find $y = -e^{2\ln(x)} = -x^2$. We easily verify that this is indeed a global solution.

Example 18. $y' = x + y$ is a DE for which the variables cannot be separated.

  No worries, very soon we will have several tools to solve this DE as well.

Existence and uniqueness of solutions

The following is a very general result that allows us to guarantee that "nice" IVPs must have a solution and that this solution is unique.

Comment. Note that any first-order DE can be written as $g(y', y, x) = 0$ where g is some function of three variables. Assuming that g is reasonable, we can solve for y' and rewrite such a DE as $y' = f(x, y)$ (for some, possibly complicated, function f).

Comment. To be precise, a solution to the IVP $y' = f(x, y)$, $y(a) = b$ is a function $y(x)$, defined on an interval I containing a, such that $y'(x) = f(x, y(x))$ for all $x \in I$ and $y(a) = b$.

Theorem 19. (existence and uniqueness) Consider the IVP $y' = f(x, y)$, $y(a) = b$.

If both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous [in a rectangle] around (a, b), then the IVP has a unique solution in some interval $x \in (a - \delta, a + \delta)$ where $\delta > 0$.

Comment. The interval around a might be very small. In other words, the δ in the theorem could be very small.

Comment. Note that the theorem makes two important assertions. First, it says that there exists a **local solution**. Second, it says that this solution is unique. These two parts of the theorem are famous results usually attributed to Peano (existence) and Picard–Lindelöf (uniqueness).

Advanced comment. The condition about $\frac{\partial}{\partial y} f(x, y)$ is a bit technical (and not optimal). If we drop this condition, we still get existence but, in general, no longer uniqueness.

Advanced comment. The interval in which the solution is unique could be smaller than the interval in which it exists. In other words, it is possible that, away from the initial condition, the solution "forks" into two or more solutions. Note that this does not contradict the theorem because it only guarantees uniqueness on a small interval.

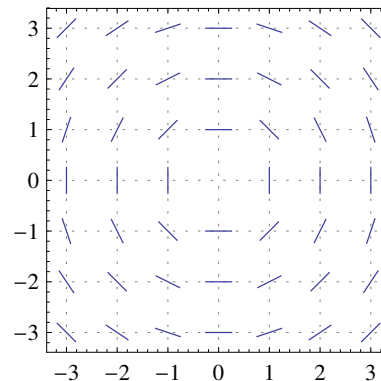**Example 20.** Consider, again, the IVP $y' = -x / y$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = -x / y$.

We compute that $\frac{\partial}{\partial y} f(x, y) = x / y^2$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $y \neq 0$.

Hence, if $b \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Comment. In Example 13, we found that the DE $y' = -x / y$ is solved by $y(x) = \pm\sqrt{D - x^2}$.

Assume $b > 0$ (things work similarly for $b < 0$). Then $y(x) = \sqrt{D - x^2}$ solves the IVP (we need to choose D so that $y(a) = b$) if we choose $D = a^2 + b^2$. This confirms that there exists a solution. On the other hand, uniqueness means that there can be no other solution to the IVP than this one.

What happens in the case $b = 0$?

Solution. In this case, the existence and uniqueness theorem does not guarantee anything. If $a \neq 0$, then $y(x) = \sqrt{a^2 - x^2}$ and $y(x) = -\sqrt{a^2 - x^2}$ both solve the IVP (so we certainly don't have uniqueness), however only in a weak sense: namely, both of these solutions are not valid locally around $x = a$ but only in an interval of which a is an endpoint (for instance, the IVP $y' = -x / y$, $y(2) = 0$ is solved by $y(x) = \pm\sqrt{4 - x^2}$ but both of these solutions are only valid on the interval $[-2, 2]$ which ends at 2, and neither of these solutions can be extended past 2).