Example 11. (review)

(a) Verify that $x(t) = \frac{1}{c - kt}$ is a one-parameter family of solutions to the DE $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 2$.

(c) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 0$.

Solution.

(a) We compute that $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{(c - kt)^2} \cdot (-k) = \frac{k}{(c - kt)^2}$.

On the other hand, $kx^2 = k\left(\frac{1}{c - kt}\right)^2 = \frac{k}{(c - kt)^2}$ as well. Thus, indeed, $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) We start with $x(t) = \frac{1}{c - kt}$ (which we know solves the DE for any value of c) and seek to choose c so that $x(0) = 2$.

Since $x(0) = \left[\frac{1}{c - kt}\right]_{t=0} = \frac{1}{c} \overset{!}{=} 2$, we find $c = \frac{1}{2}$.

Hence, the IVP has the (unique) solution $x(t) = \frac{1}{1/2 - kt}$.

(c) Proceeding as in the previous part, we now arrive at the impossible equation $\frac{1}{c} \overset{!}{=} 0$.

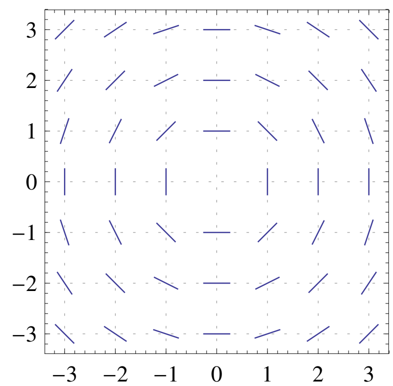
However, this suggests that we should consider taking $c \to \infty$ in $x(t) = \frac{1}{c - kt}$, which results in $x(t) = 0$.

Indeed, it is easy to verify (make sure you know what this entails!) that $x(t) = 0$ solves the IVP.

Slope fields, or sketching solutions to DEs

Example 12. Consider the DE $y' = -x/y$.

Let's pick a point, say, $(1, 2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1/2$. We therefore draw a small line through the point $(1, 2)$ with slope $-1/2$. Continuing in this fashion for several other points, we obtain the **slope field** on the right.



With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2} \frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x)$. So, yes, we actually found solutions!

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Solving DEs: Separation of variables

Example 13. Solve the DE $y' = -\frac{x}{y}$.

Solution. Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

Separate the variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$ (in particular, we are multiplying both sides by $\mathrm{d}x$).

Integrating both sides, we get $\int y\,\mathrm{d}y = \int -x\,\mathrm{d}x$.

Computing both integrals results in $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ (we combine the two constants of integration into one).

Hence $x^2 + y^2 = D$ (with $D = 2C$).

This is an **implicit form** of the solutions to the DE. We can make it explicit by solving for y. Doing so, we find $y(x) = \pm\sqrt{D - x^2}$ (choosing $+$ gives us the upper half of a circle, while the negative sign gives us the lower half).

Comment. The step above where we break $\frac{\mathrm{d}y}{\mathrm{d}x}$ apart and then integrate may sound sketchy!

However, keep in mind that, after we find a solution $y(x)$, even if by sketchy means, we can (and should!) verify that $y(x)$ is indeed a solution by plugging into the DE. We actually already did that in the previous example!

In general, **separation of variables** solves $y' = g(x)h(y)$ by writing the DE as $\frac{1}{h(y)}\mathrm{d}y = g(x)\mathrm{d}x$.

Note that $\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$ is indeed equivalent to $\int \frac{1}{h(y)}\mathrm{d}y = \int g(x)\mathrm{d}x + C$. Why?! (Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to the integrals...)

Example 14. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$.

Comment. Instead of using what we found in the previous example, we start from scratch to better illustrate the solution process (and how we can use the initial condition right away to determine the value of the constant of integration).

Solution. We separate variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$.

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, and we use $y(0) = -3$ to find $\frac{1}{2}(-3)^2 = 0 + C$ so that $C = \frac{9}{2}$.

Hence, $x^2 + y^2 = 9$ is an **implicit** form of the solution.

Solving for y, we get $y = -\sqrt{9 - x^2}$ (note that we have to choose the negative sign so that $y(0) = -3$).

Comment. Note that our solution is a **local solution**, meaning that it is valid (and solves the DE) locally around $x = 0$ (from the initial condition). However, it is not a **global solution** because it doesn't make sense outside of x in the interval $[-3, 3]$.

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