Example 5. Verify that $e^y y' = 1$ is solved by $y(x) = \ln(x + C)$.

Solution. $y'(x) = \frac{1}{x + C}$ and $e^{y(x)} = x + C$.

Hence, $e^y y' = (x + C)\frac{1}{x + C} = 1$.

Because $y(x)$ solves the DE for any value of the parameter C, we say that $y(x) = \ln(x + C)$ is a **one-parameter family** of solutions to the DE.

Example 6. Consider the DE $y'' = y' + 6y$. For which r is e^{rx} a solution?

Solution. If $y(x) = e^{rx}$, then $y'(x) = r e^{rx}$ and $y''(x) = r^2 e^{rx}$.

Plugging $y(x) = e^{rx}$ into the DE, we get $r^2 e^{rx} = r e^{rx} + 6e^{rx}$ which simplifies to $r^2 = r + 6$.

This has the two solutions $r = -2$, $r = 3$. Hence e^{-2x} and e^{3x} are solutions of the DE.

In fact, we check that $Ae^{-2x} + Be^{3x}$ is a **two-parameter family** of solutions to the DE.

Important comment. It is no coincidence that the order of the DE is 2, whereas the previous example has order 1. In general, we expect a DE of order r to have a solution with r parameters.

Example 7. Solve the DE $y' = x^2 + x$.

Solution. Note that the DE simply asks for a function $y(x)$ with a specific derivative (in particular, the right-hand side does not involve $y(x)$). In other words, the desired $y(x)$ is an **antiderivative** of $x^2 + x$. We know from Calculus II that we can find antiderivatives by integrating:

$$y(x) = \int (x^2 + x)\mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Moreover, we know from Calculus II that there are no other solutions. In other words, we found the **general solution** to the DE.

To single out a **particular solution**, we need to specify additional conditions (typically one condition per parameter in the general solution). For instance, it is common to impose **initial conditions** such as $y(1) = 2$. A DE together with an initial condition is called an **initial value problem** (IVP).

Example 8. Solve the IVP $y' = x^2 + x$ with $y(1) = 2$.

Solution. From the previous example, we know that $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$.

Since $y(1) = \frac{1}{3} + \frac{1}{2} + C = \frac{5}{6} + C \overset{!}{=} 2$, we find $C = 2 - \frac{5}{6} = \frac{7}{6}$.

Hence, $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{7}{6}$ is the (unique) solution of the IVP.

Example 9. Solve the DE $y'' = x^2 + x$.

Solution. We now take two antiderivatives of $x^2 + x$ to get

$$y(x) = \iint (x^2 + x)\mathrm{d}x\mathrm{d}x = \int \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + C\right)\mathrm{d}x = \frac{1}{12}x^4 + \frac{1}{6}x^3 + Cx + D,$$

where it is important that we give the second constant of integration a name different from the first.

Again, this is the **general solution** to the DE. The DE is of order 2 and, as expected, the general solution has 2 parameters.

Important. Note that we are working with functions $y(x)$ of a single variable. This allows us to write simply y' for $\frac{\mathrm{d}}{\mathrm{d}x}y(x)$ without risk of confusion.

Of course, we may use different variables such as $x(t)$ and $x' = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, as long as this is clear from the context.

Differential equations that involve only derivatives with respect to a single variable are known as **ordinary differential equations** (ODEs).

On the other hand, differential equations that involve derivatives with respect to several variables are referred to as **partial differential equations** (PDEs).

Example 10. The DE

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 u(x, y) + \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^2 u(x, y) = 0,$$

often abbreviated as $u_{xx} + u_{yy} = 0$, is a partial differential equation in two variables.

This particular PDE is known as **Laplace's equation** and describes, for instance, steady-state heat distributions.
https://en.wikipedia.org/wiki/Laplace%27s_equation

This and other fundamental PDEs will be discussed in Differential Equations II.

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