Notes for Lecture 1

Tue, 8/16/2022

Review: Computing derivatives

Given a function $y(x)$, we learned in Calculus I that its **derivative**

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(where $\Delta y = y(x + \Delta x) - y(x)$) has the following two important characterizations:

- $y'(x)$ is the **slope of the tangent line** of the graph of $y(x)$ at x, and
- $y'(x)$ is the **rate of change** of $y(x)$ at x.

Moreover, we learned simple rules to compute the derivative of functions:

- **(sum rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = f'(x) + g'(x)$
- **(product rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- **(chain rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$

  Comment. If we write $t = g(x)$ and $y = f(t)$, then the chain rule takes the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$. In other words, the chain rule expresses the fact that we can treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ (which initially is just a notation for $y'(x)$) as an honest fraction.

- **(basic functions)** $\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} \sin(x) = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} \cos(x) = -\sin(x)$

These rules are enough to compute the derivative of any function that we can build from the basic functions using algebraic operations and composition. On the other hand, as you probably recall from Calculus II, reversing the operation of differentiation (i.e. computing antiderivatives) is much more difficult.

In particular, there exist simple functions (such as e^{x^2}) whose antiderivative cannot be expressed in terms of the basic functions above.

Example 1. Derive the **quotient rule** from the rules above.

Solution. We write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ and apply the product rule to get

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} + f(x) \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)}.$$

By the chain rule combined with $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x} = -\frac{1}{x^2}$, we have $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)} = -\frac{1}{g(x)^2} g'(x)$. Using this in the previous formula,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} - f(x) \frac{1}{g(x)^2} g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}.$$

Putting the final two fractions on a common denominator, we obtain the familiar quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

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First examples of differential equations

Example 2. If $y(x)=e^{x^2}$ then $y'(x)=2xe^{x^2}=2xy(x)$ or, for short, $y'=2xy$.

Accordingly, we say that $y(x)=e^{x^2}$ is a **solution** to the **differential equation** (DE) $y'=2xy$.

Example 3. By computing its derivative, determine a DE solved by $y(x)=\sin(3x)$.

Solution. $y'(x)=3\cos(3x)=3\sqrt{1-(\sin(3x))^2}=3\sqrt{1-y(x)^2}$

[Here we used that $\cos(x)^2+\sin(x)^2=1$, which implies that $\cos(x)=\sqrt{1-\sin(x)^2}$.]

Hence, $y(x)=\sin(3x)$ solves the differential equation $y'=3\sqrt{1-y^2}$.

Example 4. By computing its second derivative, determine another DE solved by $y(x)=\sin(3x)$.

Solution. $y''(x)=-9\sin(3x)=-9y(x)$.

Thus, $y(x)=\sin(3x)$ also solves the differential equation $y''=-9y$.

If the highest derivative appearing in a DE is an rth derivative, we say that the DE has **order** r.

For instance. The DE $y'=3\sqrt{1-y^2}$ has order 1 (such DEs are also called first order DEs).

On the other hand, the DE $y''=-9y$ has order 2 (such DEs are also called second order DEs).

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