Final: practice

MATH 238 — Differential Equations I
final exam: Thursday, Dec 8

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms.

(b) Retake both midterm exams.

(c) Do the problems below. (Solutions are posted.)

Problem 2. The position $y(t)$ of a certain mass on a spring is described by $2y'' + dy' + 3y = F\sin(4\omega t)$.

(a) Assume first that there is no external force, i.e. $F = 0$. For which values of d is the system overdamped?

(b) Now, $F \neq 0$ and the system is undamped, i.e. $d = 0$. For which values of ω, if any, does resonance occur?

Solution.

(a) The discriminant of the characteristic equation is $d^2 - 24$. Hence the system is overdamped if $d^2 - 24 > 0$, that is $d > \sqrt{24} = 2\sqrt{6}$.

(b) The natural frequency is $\sqrt{\frac{3}{2}}$. Resonance therefore occurs if $4\omega = \sqrt{\frac{3}{2}}$ or, equivalently, $\omega = \frac{1}{4}\sqrt{\frac{3}{2}}$.

Problem 3.

(a) Determine the Laplace transform $\mathcal{L}(2e^{4t} - 6e^{-t} + 3)$.

(b) Determine the Laplace transform $\mathcal{L}((2t-5)e^{-3t})$.

(c) Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{s-16}{s^2-2s-8}\right)$.

Solution.

(a) $\mathcal{L}(2e^{4t} - 6e^{-t} + 3) = 2\mathcal{L}(e^{4t}) - 6\mathcal{L}(e^{-t}) + 3\mathcal{L}(1) = \frac{2}{s-4} - \frac{6}{s+1} + \frac{3}{s}$

(b) $\mathcal{L}((2t-5)e^{-3t}) = 2\mathcal{L}(te^{-3t}) - 5\mathcal{L}(e^{-3t}) = \frac{2}{(s+3)^2} - \frac{5}{s+3}$

Here, we combined $\mathcal{L}(tf(t)) = -F'(s)$ with $\mathcal{L}(e^{-3t}) = \frac{1}{s+3}$ to get $\mathcal{L}(te^{-3t}) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s+3} = \frac{1}{(s+3)^2}$.

(c) Note that $s^2 - 2s - 8 = (s+2)(s-4)$. We use partial fractions to write $\frac{s-16}{(s+2)(s-4)} = \frac{A}{s+2} + \frac{B}{s-4}$. We find the coefficients as

$$A = \left.\frac{s-16}{s-4}\right|_{s=-2} = 3, \quad B = \left.\frac{s-16}{s+2}\right|_{s=4} = -2.$$

Hence $\mathcal{L}^{-1}\left(\frac{s-16}{s^2-2s-8}\right) = \mathcal{L}^{-1}\left(\frac{3}{s+2} - \frac{2}{s-4}\right) = 3e^{-2t} - 2e^{4t}$.

Problem 4. Determine the Laplace transform of the unique solutions. [Do not determine the solution and do not simplify.]

(a) $y'' + 4y' - 3y = 2e^{-4t} + 5t^2,\ y(0) = 7,\ y'(0) = -2$.

(b) $y_1' = 3y_1 - 4y_2 - 5t^2 e^{-3t},\ y_2' = y_1 + 2y_2,\ y_1(0) = 3,\ y_2(0) = -2$.

Solution.

(a) The DE $y'' + 4y' - 3y = 2e^{-4t} + 5t^2$ transforms into

$$s^2 Y - s y(0) - y'(0) + 4(sY - y(0)) - 3Y = (s^2 + 4s - 3)Y - 7s - 26 = \frac{2}{s+4} + \frac{10}{s^3}.$$

Accordingly, $Y(s) = \frac{1}{s^2+4s-3}\left[\frac{2}{s+4} + \frac{10}{s^3} + 7s + 26\right]$ is the Laplace transform of the unique solution to the IVP.

(b) $y_1' = 3y_1 - 4y_2 - 5t^2 e^{-3t}$ transforms into $sY_1 - \underbrace{y_1(0)}_{=3} = 3Y_1 - 4Y_2 - \frac{10}{(s+3)^3}$ or $(s-3)Y_1 + 4Y_2 = 3 - \frac{10}{(s+3)^3}$.

Likewise, $y_2' = y_1 + 2y_2$ transforms into $sY_2 - \underbrace{y_2(0)}_{=-2} = Y_1 + 2Y_2$.

By the second equation, $Y_1 = (s-2)Y_2 + 2$. Used in the first, we get $(s-3)((s-2)Y_2 + 2) + 4Y_2 = 3 - \frac{10}{(s+3)^3}$ or $((s-3)(s-2) + 4)Y_2 = 3 - \frac{10}{(s+3)^3} - 2(s-3)$.

Hence, $Y_2 = \left(3 - \frac{10}{(s+3)^3} - 2(s-3)\right) / ((s-3)(s-2) + 4)$.

Correspondingly, $Y_1 = (s-2)Y_2 + 2 = (s-2)\left(3 - \frac{10}{(s+3)^3} - 2(s-3)\right) / ((s-3)(s-2) + 4) + 2$.

Comment. The important thing is to realize that, after taking the Laplace transform of the two differential equations, we obtain two ordinary (linear) equations in Y_1 and Y_2 that we can easily solve (although the formulas become involved). If we wanted to compute the solutions $y_i(t)$, we could then do a partial fraction decomposition of the rational functions for Y_i (both the involved roots would be irrational and the result not pretty).

For the final exam, you will be provided the following table of Laplace transforms.

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at} f(t)$	$F(s-a)$
$t f(t)$	$-F'(s)$
$u_a(t) f(t-a)$	$e^{-sa} F(s)$

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