Example 107. (review) The PNT estimates that there are how many primes up to 2^{1024}? What is the approximate proportion of primes among numbers up to 2^{1024}?

Solution. By the PNT (applied with $x = 2^{1024}$), there are approximately $\frac{x}{\ln(x)} = \frac{2^{1024}}{\ln(2^{1024})} = \frac{2^{1024}}{1024\ln(2)} \approx \frac{2^{1024}}{709.8}$ primes up to $x = 2^{1024}$. In particular, the proportion of primes up to 2^{1024} is about $\frac{1}{709.8}$.
That means, roughly, 1 in 710 numbers of this magnitude are prime.

Example 108. Using Sage, determine all numbers n up to 5000, for which 2 is a Fermat liar.

Solution. Recall that, if N is composite, then a residue a is a Fermat liar modulo N if $a^{N-1} \equiv 1 \pmod{N}$.

```
Sage] def is_fermat_liar(x, n):
          return not is_prime(n) and power_mod(x, n-1, n) == 1

Sage] [ n for n in [1..5000] if is_fermat_liar(2, n) ]
```

$$[341, 561, 645, 1105, 1387, 1729, 1905, 2047, 2465, 2701, 2821, 3277, 4033, 4369, 4371, 4681]$$

Even if you have never written any code, you can surely figure out what's going on!
Heads-up! The improved primality test discussed today will reduce this list to just $2047, 3277, 4033, 4681$.

The Miller–Rabin primality test

Review. The congruence $x^2 \equiv 1 \pmod{p}$ has only the solutions $x \equiv \pm 1$.
By contrast, if n is composite (and odd), then $x^2 \equiv 1 \pmod{n}$ has additional solutions.

The Miller–Rabin primality test exploits this difference to fix the issues of the Fermat primality test.

The Fermat primality test picks a and checks whether $a^{n-1} \equiv 1 \pmod{n}$.

- If $a^{n-1} \not\equiv 1 \pmod{n}$, then we are done because n is definitely not a prime.
- If $a^{n-1} \equiv 1 \pmod{n}$, then either n is prime or a is a Fermat liar.
  But instead of leaving off here, we can dig a little deeper:
  Note that $a^{(n-1)/2}$ satisfies $x^2 \equiv 1 \pmod{n}$. If n is prime, then $x \equiv \pm 1$ so that $a^{(n-1)/2} \equiv \pm 1 \pmod{n}$.
  - Hence, if $a^{(n-1)/2} \not\equiv \pm 1 \pmod{n}$, then we again know for sure that n is not a prime.
    Advanced comment. In fact, can you see that we can now factor n? See below.
  - If $a^{(n-1)/2} \equiv 1 \pmod{n}$ and $\frac{n-1}{2}$ is divisible by 2, we continue and look at $a^{(n-1)/4} \pmod{n}$.
    - If $a^{(n-1)/4} \not\equiv \pm 1 \pmod{n}$, then n is not a prime.
    - If $a^{(n-1)/4} \equiv 1 \pmod{n}$ and $\frac{n-1}{4}$ is divisible by 2, we continue...

Write $n - 1 = 2^s \cdot m$ with m odd. In conclusion, if n is a prime, then

$$a^m \equiv 1 \quad \text{or, for some } r = 0, 1, \ldots, s-1, \quad a^{2^r m} \equiv -1 \pmod{n}.$$

In other words, if n is a prime, then the values $a^m, a^{2m}, \ldots, a^{2^s m}$ must be of the form $1, 1, \ldots, 1$ or $\ldots, -1, 1, 1, \ldots, 1$. If the values are of this form even though n is composite, then a is a **strong liar** modulo n.

This gives rise to the following improved primality test:

Miller–Rabin primality test

Input: number n and parameter k indicating the number of tests to run

Output: "not prime" or "likely prime"

Algorithm:

Write $n-1=2^s\cdot m$ with m odd.
Repeat k times:
  Pick a random number a from $\{2,3,\ldots,n-2\}$.
  If $a^m\not\equiv 1 \pmod n$ and $a^{2^r m}\not\equiv -1 \pmod n$ for all $r=0,1,\ldots,s-1$, then
    stop and output "not prime".
Output "likely prime".

Comment. If n is composite, then fewer than a quarter of the values for a can possibly be strong liars. In other words, for all composite numbers, the odds that the Miller–Rabin test returns "likely prime" are less than 4^{-k}.

Comment. Note that, though it looks more involved, the Miller–Rabin test is essentially as fast as the Fermat primality test (recall that, to compute a^{n-1}, we proceed using binary exponentiation).

Advanced comments. This is usually implemented as a probabilistic test. However, assuming GRH (the generalized Riemann hypothesis), it becomes a deterministic algorithm if we check $a=2,3,\ldots,\lfloor 2(\log n)^2\rfloor$. This is mostly of interest for theoretical applications. For instance, this then becomes a polynomial time algorithm for checking whether a number is prime.

More recently, in 2002, the AKS primality test was devised. This test is polynomial time (without relying on outstanding conjectures like GRH).

Example 109. Suppose we want to determine whether $n=221$ is a prime. Simulate the Miller–Rabin primality test for the choices $a=24$, $a=38$ and $a=47$.

Solution. $n-1=4\cdot 55=2^s\cdot m$ with $s=2$ and $m=55$.

- For $a=24$, we compute $a^m=24^{55}\equiv 80\not\equiv\pm 1 \pmod{221}$. We continue with $a^{2m}\equiv 80^2\equiv 212\not\equiv -1$, and conclude that n is not a prime.

  Note. We do not actually need to compute that $a^{n-1}=a^{4m}\equiv 81$, which features in the Fermat test and which would also lead us to conclude that n is not prime.

- For $a=38$, we compute $a^m=38^{55}\equiv 64\not\equiv\pm 1 \pmod{221}$. We continue with $a^{2m}\equiv 64^2\equiv 118\not\equiv -1$ and conclude that n is not a prime.

  Note. This case is somewhat different from the previous in that 38 is a Fermat liar. Indeed, $a^{4m}\equiv 118^2\equiv 1 \pmod{221}$. This means that we have found a nontrivial sqareroot of 1. In this case, the Fermat test would have failed us while the Miller–Rabin test succeeds.

- For $a=47$, we compute $a^m=47^{55}\equiv 174\not\equiv\pm 1 \pmod{221}$. We continue with $a^{2m}\equiv 174^2\equiv -1$. We conclude that n is a prime or a is a strong liar. In other words, we are not sure but are (incorrectly) leaning towards thinking that 221 was likely a prime.

Comment. In this example, only 4 of the 218 residues $2,3,\ldots,219$ are strong liars (namely $21,47,174,200$). For comparison, there are 14 Fermat liars (namely $18,21,38,47,64,86,103,118,135,157,174,183,200,203$). [Note that ± 1 are Fermat as well as strong liars, too. However, these are usually excluded when testing.]

Example 110. Show that, if $x^2 \equiv 1 \pmod{n}$ but $x \not\equiv \pm 1 \pmod{n}$, then we can find a factor of n.

Show that, if $a^{n-1} \equiv 1 \pmod{n}$ but $a^{(n-1)/2} \not\equiv \pm 1 \pmod{n}$, then we can find a factor of n.

Solution. More generally, suppose we have $x^2 \equiv 1 \pmod{n}$ but $x \not\equiv \pm 1 \pmod{n}$ (that's what we have with $x = a^{(n-1)/2}$). Note that $x^2 - 1 = (x-1)(x+1) \equiv 0 \pmod{n}$. Since n divides the product of $(x-1)$ and $(x+1)$ but neither of them individually, it follows that $\gcd(x-1, n)$ is a proper divisor of n.

Comment. This happens within Miller–Rabin if we have, for instance, if $a^{n-1} \equiv 1 \pmod{n}$ but $a^{(n-1)/2} \not\equiv \pm 1 \pmod{n}$ (take $x = a^{(n-1)/2}$). However, note that this only happens if a is a Fermat liar modulo n, and these are typically very rare. So, unfortunately, we have not discovered an efficient factorization algorithm. [But we have run into an idea which is used for some of the best known factorization algorithms. If time permits, more on that later...]

For instance. In the case of $a = 38$ and $n = 221$ in Example 109 we found that $118^2 \equiv 1 \pmod{221}$. It follows that $\gcd(118-1, 221) = 13$ (as well as $\gcd(118+1, 221) = 17$) are divisors of $n = 221$.

Comment. Explain that the same approach works for any quadratic equation $x^2 \equiv a^2 \pmod{n}$. Or even any quadratic equation for which we have more than two solutions.

Example 111. In Example 96, we saw that all $\phi(561) = 320$ invertible residues a modulo 561 are Fermat liars (that is, they all satisfy $a^{560} \equiv 1 \pmod{561}$). How many strong liars are there?

Solution. There are 10 strong liars in total: $\pm 1, \pm 50, \pm 101, \pm 103, \pm 256$.

In particular, only 8 of the 558 residues $2, 3, \ldots, 559$ are strong liars. That's about 1.43% (much less than the theoretic bound of 25%).

Advanced comment. Among numbers N up to 1000, the proportion of strong liars (taken as the number of strong liars among the residues $2, 3, \ldots, N-2$) is highest for $N = 703$ in which case we have 160 strong liars among $2, 3, \ldots, 701$ (which is a proportion of $\frac{160}{700} = \frac{8}{35} \approx 22.86\%$, just shy of the theoretical bound of 25%.).

Here (as illustrated in the case of 561 above) we define the proportion of strong liars to be the proportion of residues among $2, 3, \ldots, N-2$, which are strong liars. In other words, we are excluding the residues ± 1 as bases. If we included these, then $N = 9$ would have the highest proportion, namely exactly 25%, because ± 1 are strong liars.