# Notes for Lecture 8

**Review.**

- A **pseudorandom generator** (PRG) takes a seed $x_0$ and produces a stream $\mathrm{PRG}(x_0) = x_1 x_2 x_3 \ldots$ of numbers, which should "look like" random numbers.

  For cryptographic purposes, these numbers should be indistinguishable from random numbers. Even for somebody who knows everything about the PRG except the seed. (See Example 57.)

- Once we have a PRG, we can use it as a **stream cipher**: Using the key $k$, we encrypt $E_k(m) = m \oplus \mathrm{PRG}(k)$. [Here, the key stream $\mathrm{PRG}(k)$ is assumed to be in bits.]

  As with the one-time pad, we must never reuse the same keystream!

- To reuse the key, we can use a **nonce**: $E_k(m) = m \oplus \mathrm{PRG}((\mathrm{nonce}, k))$, where the seed is produced by combining the $\mathrm{nonce}$ and $k$ (for instance, just concatenating them).

  The nonce is then passed (unencrypted) along with the message.

  To never reuse the same keystream, we must never use the same nonce with the same key.

## Linear feedback shift registers

Here is another basic idea to generate pseudorandom numbers:

**(linear feedback shift register (LFSR)** Let $\ell$ and $c_1, c_2, \ldots, c_\ell$ be chosen parameters.

From the seed $(x_1, x_2, \ldots, x_\ell)$, where each $x_i$ is one bit, we produce the sequence

$$x_{n+\ell} \equiv c_1 x_{n+\ell-1} + c_2 x_{n+\ell-2} + \ldots + c_\ell x_n \pmod{2}.$$

This method is particularly easy to implement in hardware (see Example 55), and hence suited for applications that value speed over security (think, for instance, encrypted television).

**Example 54.** Which sequence is generated by the LFSR $x_{n+2} \equiv x_{n+1} + x_n \pmod{2}$, starting with the seed $(x_1, x_2) = (0, 1)$?

**Solution.** $(x_1, x_2, x_3, \ldots) = (0, 1, 1, 0, 1, 1, \ldots)$ has period $3$.

**Note.** Observe that the two previous values determine the state, so there are $2^2 = 4$ states of the LFSR. The state $(0, 0)$ is special (it generates the zero sequence $(0, 0, 0, 0, \ldots)$), so there are $3$ other states. Hence, it is clear that the generated sequence has to repeat after at most $3$ terms.

**Comment.** Of course, if we don't reduce modulo $2$, then the sequence $x_{n+2} = x_{n+1} + x_n$ generates the Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, \ldots$

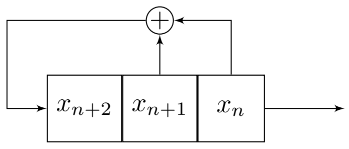
**Example 55.** Which sequence is generated by the LFSR $x_{n+3} \equiv x_{n+1} + x_n \pmod{2}$, starting with the seed $(x_1, x_2, x_3) = (0, 0, 1)$? What is the period?

[Let us first note that the LFSR has $2^3 = 8$ states. Since the state $(0, 0, 0)$ remains zero forever, $7$ states remain. This means that the generated sequence must be periodic, with period at most $7$.]

**Solution.** $(x_1, x_2, x_3, \ldots) = (0, 0, 1, 0, 1, 1, 1, 0, 0, 1, \ldots)$ has period $7$.

Again, this is not surprising: $3$ previous values determine the state, so there are $2^3 = 8$ states. The state $(0, 0, 0)$ is special, so there are $7$ other states.

Note that this LFSR can be implemented in hardware using three registers (labeled $x_n, x_{n+1}, x_{n+2}$ in the sketch to the right). During each cycle, the value of $x_n$ is read off as the next value produced by the LFSR.



**Note.** In the part $0, 0, 1, 0, 1, 1, 1$ that repeats, the bit $1$ occurs more frequently than $0$.

The reason for that is that the special state $(0, 0, 0)$ cannot appear.

For the same reason, the bit $1$ will always occur slightly more frequently than $0$ in LFSRs. However, this becomes negligible if the period is huge, like $2^{31} - 1$ in Example 56.

**Example 56.** The recurrence $x_{n+31} \equiv x_{n+28} + x_n \pmod{2}$, with a nonzero seed, generates a sequence that has period $2^{31} - 1$.

Note that this is the maximal possible period: this LFSR has $2^{31}$ states. Again, the state $(0, 0, \ldots, 0)$ is special (the entire sequence will be zero), so that there are $2^{31} - 1$ other states. This means that the terms must be periodic with period at most $2^{31} - 1$.

**Comment.** glibc (the second implementation) essentially uses this LFSR.

**Advanced comment.** One can show that, if the characteristic polynomial $f(T) = x^\ell + c_1 x^{\ell-1} + c_2 x^{\ell-2} + \ldots + c_\ell$ is irreducible modulo $2$, then the period divides $2^\ell - 1$. Here, $f(T) = T^{31} + T^{28} + 1$ is irreducible modulo $2$, so that the period divides $2^{31} - 1$. However, $2^{31} - 1$ is a prime, so that the period must be exactly $2^{31} - 1$.

**Example 57.** Eve intercepts the ciphertext $c = (1111\ 1011\ 0000)_2$ from Alice to Bob. She knows that the plaintext begins with $m = (1100\ 0\ldots)_2$. Eve thinks a stream cipher using a LFSR with $x_{n+3} \equiv x_{n+2} + x_n \pmod{2}$ was used. If that's the case, what is the plaintext?

**Solution.** The initial piece of the keystream is $\mathrm{PRG} = m \oplus c = (1100\ 0\ldots)_2 \oplus (1111\ 1\ldots)_2 = (0011\ 1\ldots)_2$.

Each $x_n$ is a single bit, and we have $x_1 \equiv 0$, $x_2 \equiv 0$, $x_3 \equiv 1$. The given LFSR produces $x_4 \equiv x_3 + x_1 \equiv 1$, $x_5 \equiv x_4 + x_2 \equiv 1$, $x_6 \equiv 0$, $x_7 \equiv 1$, and so on. Continuing, we obtain $\mathrm{PRG} = x_1 x_2 \ldots = (0011\ 1010\ 0111)_2$.

Hence, the plaintext would be $m = c \oplus \mathrm{PRG} = (1111\ 1011\ 0000)_2 \oplus (0011\ 1010\ 0111)_2 = (1100\ 0001\ 0111)_2$.

A PRG is **predictable** if, given the stream it outputs (but not the seed), one can with some precision predict what the next bit will be (i.e. do better than just guessing the next bit).

In other words: the bits generated by the PRG must be indistinguishable from truly random bits, even in the eyes of someone who knows everything about the PRG except the seed.

The PRGs we discussed so far are entirely predictable because the state of the PRGs is part of the random stream they output.

For instance, for a given LFSR, it is enough to know any $\ell$ consecutive outputs $x_n, x_{n+1}, \ldots, x_{n+\ell-1}$ in order to predict all future output.