Review. If N is composite, then a residue a is a Fermat liar modulo N if $a^{N-1} \equiv 1 \pmod{N}$.

Example 104. Using Sage, determine all numbers n up to 5000, for which 2 is a Fermat liar.

```
Sage] def is_fermat_liar(x, n):
          return not is_prime(n) and power_mod(x, n-1, n) == 1

Sage] [ n for n in [1..5000] if is_fermat_liar(2, n) ]
```

$[341, 561, 645, 1105, 1387, 1729, 1905, 2047, 2465, 2701, 2821, 3277, 4033, 4369, 4371, 4681]$

Even if you have never written any code, you can surely figure out what's going on!

Heads-up! The improved primality test discussed today will reduce this list to just $2047, 3277, 4033, 4681$.

The Miller–Rabin primality test

Review. The congruence $x^2 \equiv 1 \pmod{p}$ has only the solutions $x \equiv \pm 1$.
By contrast, if n is composite (and odd), then $x^2 \equiv 1 \pmod{n}$ has additional solutions.

The Miller–Rabin primality test exploits this difference to fix the issues of the Fermat primality test.

The Fermat primality test picks a and checks whether $a^{n-1} \equiv 1 \pmod{n}$.

- If $a^{n-1} \not\equiv 1 \pmod{n}$, then we are done because n is definitely not a prime.
- If $a^{n-1} \equiv 1 \pmod{n}$, then either n is prime or a is a Fermat liar.
  But instead of leaving off here, we can dig a little deeper:
  Note that $a^{(n-1)/2}$ satisfies $x^2 \equiv 1 \pmod{n}$. If n is prime, then $x \equiv \pm 1$ so that $a^{(n-1)/2} \equiv \pm 1 \pmod{n}$.
  - Hence, if $a^{(n-1)/2} \not\equiv \pm 1 \pmod{n}$, then we again know for sure that n is not a prime.
    Advanced comment. In fact, we can now factor n! See bonus challenge below.
  - If $a^{(n-1)/2} \equiv 1 \pmod{n}$ and $\frac{n-1}{2}$ is divisible by 2, we continue and look at $a^{(n-1)/4} \pmod{n}$.
    - If $a^{(n-1)/4} \not\equiv \pm 1 \pmod{n}$, then n is not a prime.
    - If $a^{(n-1)/4} \equiv 1 \pmod{n}$ and $\frac{n-1}{4}$ is divisible by 2, we continue...

Write $n - 1 = 2^s \cdot m$ with m odd. In conclusion, if n is a prime, then

$$a^m \equiv 1 \quad \text{or, for some } r = 0, 1, \ldots, s-1, \quad a^{2^r m} \equiv -1 \pmod{n}.$$

In other words, if n is a prime, then the values $a^m, a^{2m}, \ldots, a^{2^s m}$ must be of the form $1, 1, \ldots, 1$ or $\ldots, -1, 1, 1, \ldots, 1$. If the values are of this form even though n is composite, then a is a **strong liar** modulo n.

This gives rise to the following improved primality test:

Miller–Rabin primality test

Input: number n and parameter k indicating the number of tests to run
Output: "not prime" or "likely prime"
Algorithm:

Write $n - 1 = 2^s \cdot m$ with m odd.
Repeat k times:
  Pick a random number a from $\{2, 3, \ldots, n-2\}$.
  If $a^m \not\equiv 1 \pmod{n}$ and $a^{2^r m} \not\equiv -1 \pmod{n}$ for all $r = 0, 1, \ldots, s-1$, then
    stop and output "not prime".
Output "likely prime".

Comment. If n is composite, then fewer than a quarter of the values for a can possibly be strong liars. In other words, for all composite numbers, the odds that the Miller–Rabin test returns "likely prime" are less than 4^{-k}.

Comment. Note that, though it looks more involved, the Miller–Rabin test is essentially as fast as the Fermat primality test (recall that, to compute a^{n-1}, we proceed using binary exponentiation).

Advanced comments. This is usually implemented as a probabilistic test. However, assuming GRH (the generalized Riemann hypothesis), it becomes a deterministic algorithm if we check $a = 2, 3, \ldots, \lfloor 2(\log n)^2 \rfloor$. This is mostly of interest for theoretical applications. For instance, this then becomes a polynomial time algorithm for checking whether a number is prime.

More recently, in 2002, the AKS primality test was devised. This test is polynomial time (without relying on outstanding conjectures like GRH).

Example 105. Suppose we want to determine whether $n = 221$ is a prime. Simulate the Miller–Rabin primality test for the choices $a = 24$, $a = 38$ and $a = 47$.

Solution. $n - 1 = 4 \cdot 55 = 2^s \cdot m$ with $s = 2$ and $m = 55$.

- For $a = 24$, we compute $a^m = 24^{55} \equiv 80 \not\equiv \pm 1 \pmod{221}$. We continue with $a^{2m} \equiv 80^2 \equiv 212 \not\equiv -1$, and conclude that n is not a prime.

  Note. We do not actually need to compute that $a^{n-1} = a^{4m} \equiv 81$, which features in the Fermat test and which would also lead us to conclude that n is not prime.

- For $a = 38$, we compute $a^m = 38^{55} \equiv 64 \not\equiv \pm 1 \pmod{221}$. We continue with $a^{2m} \equiv 64^2 \equiv 118 \not\equiv -1$ and conclude that n is not a prime.

  Note. This case is somewhat different from the previous in that 38 is a Fermat liar. Indeed, $a^{4m} \equiv 118^2 \equiv 1 \pmod{221}$. This means that we have found a nontrivial sqareroot of 1. In this case, the Fermat test would have failed us while the Miller–Rabin test succeeds.

- For $a = 47$, we compute $a^m = 47^{55} \equiv 174 \not\equiv \pm 1 \pmod{221}$. We continue with $a^{2m} \equiv 174^2 \equiv -1$. We conclude that n is a prime or a is a strong liar. In other words, we are not sure but are (incorrectly) leaning towards thinking that 221 was likely a prime.

Comment. In this example, only 4 of the 218 residues $2, 3, \ldots, 219$ are strong liars (namely $21, 47, 174, 200$). For comparison, there are 14 Fermat liars (namely $18, 21, 38, 47, 64, 86, 103, 118, 135, 157, 174, 183, 200, 203$).

Example 106. In Example 98, we saw that all $\phi(561) = 320$ invertible residues a modulo 561 are Fermat liars (that is, they all satisfy $a^{560} \equiv 1 \pmod{561}$). How many of them are strong liars?

Solution. Only 8 of the 558 residues $2, 3, \ldots, 559$ are strong liars (namely $50, 101, 103, 256, 305, 458, 460, 511$). That's about 1.43% (much less than the theoretic bound of 25%).

(bonus challenge) For which $N < 1000$ is the proportion of strong liars the highest?

Here (as illustrated in the case of 561 above) we define the proportion of strong liars to be the proportion of residues among $2, 3, \ldots, N-2$, which are strong liars.

[That proportion is almost 23%, just shy of the theoretical bound of 25%.]

Send in a solution by next week for a bonus point!

Extra excursion on Mersenne primes

Example 107. In 12/2018, a new largest (proven) prime was found: $2^{82,589,933}-1$.

https://www.mersenne.org/primes/?press=M82589933

This is a **Mersenne prime** (like the last 17 record primes). It has a bit over 24.8 million (decimal) digits (versus 23.2 for the previous record). The prime was found as part of GIMPS (Great Internet Mersenne Prime Search), which offers a $3,000 award for each new Mersenne prime discovered.

The EFF (Electronic Frontier Foundation) is offering $150,000 (donated anonymously for that specific purpose) for the discovery of the first prime with at least 100 million decimal digits.

https://www.eff.org/awards/coop

[Prizes of $50,000 and $100,000 for primes with 1 and 10 million digits have been claimed in 2000 and 2009.]

Definition 108. A **Mersenne prime** is a prime of the form 2^n-1.

For instance. The first few Mersenne primes have exponents $2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, \ldots$ All of these exponents are primes (but not all primes work: for instance, $2^{11}-1=23\cdot 89$). See below.

Anecdote. Euler proved in 1772 that $2^{31}-1$ is prime (then, and until 1867, the largest known prime).

"$2^{31}-1$ is probably the greatest [Mersenne prime] that ever will be discovered; for as they are merely curious, without being useful, it is not likely that any person will attempt to find one beyond it." — P. Barlow, 1811

https://en.wikipedia.org/wiki/2,147,483,647

Mersenne primes give rise precisely to all even perfect numbers (numbers whose proper divisors sum to the number itself; for instance, 6 is perfect because $6=1+2+3$). Indeed, Euclid showed that, if 2^p-1 is prime, then $2^{p-1}(2^p-1)$ is perfect [$p=2$: $2\cdot 3=6$, $p=3$: $4\cdot 7=28=1+2+4+7+14$, $p=5$: $16\cdot 33=528$, ...]. It is not known whether odd perfect numbers exist.

Example 109. (geometric sum) Evaluate $1+x+x^2+\ldots+x^n$.

Solution. $(1+x+x^2+\ldots+x^n)(x-1)=x^{n+1}-1$, so that $1+x+x^2+\ldots+x^n=\frac{x^{n+1}-1}{x-1}$.

Geometric series. In particular, $\sum_{k=1}^{\infty} x^k=\lim_{n\to\infty}\frac{x^{n+1}-1}{x-1}=\frac{1}{1-x}$, provided that $|x|<1$.

Lemma 110. If $r \mid n$, then $x^r-1 \mid x^n-1$.

Proof. Indeed, we have $x^n-1=(x^r-1)(1+x^r+x^{2r}+\ldots+x^{n-r})$.

Comment. For a tiny bit more detail, write $n=rs$. It follows from $x^s-1=(x-1)(1+x+x^2+\ldots+x^{s-1})$ that $x^{rs}-1=(x^r-1)(1+x^r+x^{2r}+\ldots+x^{r(s-1)})$. □

Corollary 111. 2^n-1 can only be prime if n is prime.

Proof. It follows from the previous lemma that, if $n=rs$ is composite, then 2^n-1 is divisible by 2^r-1 (as well as 2^s-1). □

For instance. $2^6-1=63$ is divisible by both $2^2-1=3$ and $2^3-1=7$.