Elliptic curve cryptography

The idea of Diffie–Hellman (used, for instance, in DH key exchange, ElGamal or DSA) can be carried over to algebraic structures different from multiplication modulo p.

Recall that the key idea is, starting from individual secrets x, y, to share g^x, g^y modulo p in order to arrive at the joint secret $g^{xy} \pmod{p}$. That's using multiplication modulo p.

One important example of other such algebraic structures, for which the analog of the discrete logarithm problem is believed to be difficult, are elliptic curves.

https://en.wikipedia.org/wiki/Elliptic_curve_cryptography

Comment. The main reason (apart from, say, diversification) is that this leads to a significant saving in key size and speed. Whereas, in practice, about 2048bit primes are needed for Diffie–Hellman, comparable security using elliptic curves is believed to only require about 256bits.

For a beautiful introduction by Dan Boneh, check out the presentation:
https://www.youtube.com/watch?v=4M8_Oo7lpiA

Points on elliptic curves

An **elliptic curve** is a (nice) cubic curve that can (typically) be written in the form

$$y^2 = x^3 + ax + b.$$

A point (x, y) is on the elliptic curve if it satisfies this equation. Each elliptic curve also contains the special point O ("the point at ∞"). [O will act as the neutral element when "adding points"]

Advanced comment. Sometimes it is useful (or necessary) to consider elliptic curves defined by more general cubic equations such as $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ (however, in most cases, a linear change of variables can transform this equation into the simpler form $y^2 = x^3 + ax + b$ mentioned above).

Example 216. Determine some points (x, y) on the elliptic curve E, described by

$$y^2 = x^3 - x + 9.$$

Solution. We can try some small values for x (say, $x = 0$, $x = 1$, $x = 2$, ...) and see what y needs to be in order to get a point on the elliptic curve. For instance, for $x = 1$, we get $x^3 - x + 9 = 9$ which implies that $(1, \pm 3)$ are points on the elliptic curve.

Doing so, we find the integral points $(0, \pm 3), (\pm 1, \pm 3)$.

On the other hand, for $x = 2$, we get $x^3 - x + 9 = 15$ which implies that $(2, \pm\sqrt{15})$ are points on the elliptic curve. However, for cryptographic purposes, we are usually not interested in such irrational points.

Much less obvious rational points include $(35, 207)$ or $\left(\frac{1}{36}, \frac{647}{216}\right)$ (see Example 218).

Comment. In general, it is a very difficult problem to determine all rational points on an elliptic curve, and lots of challenges remain open in that arena.

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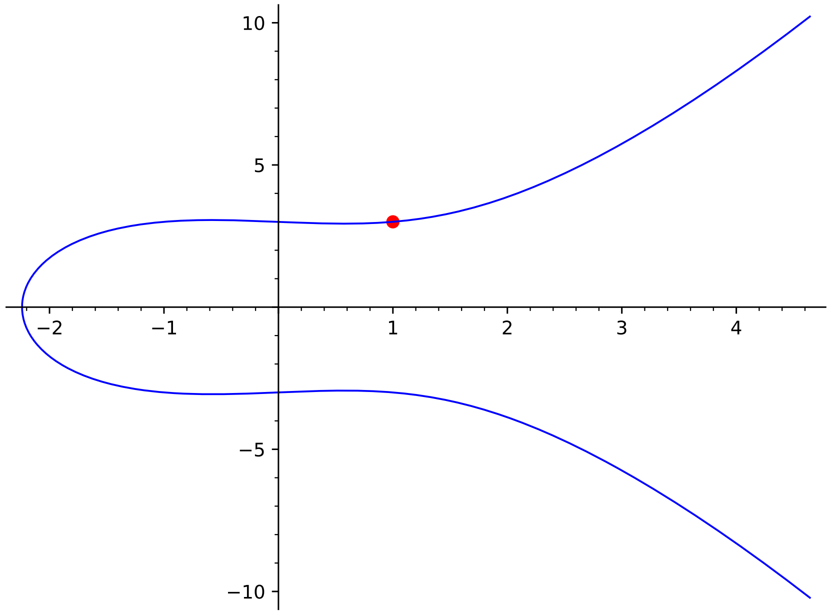
Example 217. Plot the elliptic curve E, described by $y^2 = x^3 - x + 9$ and mark the point $(1, 3)$.

Solution. We let Sage do the work for us:

```
>>> E = EllipticCurve([-1,9])
>>> E.plot() + E(1,3).plot(pointsize=50, rgbcolor=(1,0,0))
```



Adding points on elliptic curves

Note. Simply adding the coordinates of two points P and Q on an elliptic curve will (almost always) not result in a third point on the elliptic curve. However, we will define a more fancy "addition" of points, which we will denote $P \boxplus Q$, such that the $P \boxplus Q$ is on the elliptic curve as well.

Given a point $P = (x, y)$ on E, we define $-P = (x, -y)$ which is another point on E.

Let us introduce an operation $\boxplus$ in the following geometric fashion: given two points P, Q, the line through these two points intersects the curve in a third point R.

We then define $P \boxplus Q = -R$.

We remark that $P \boxplus (-P)$ is the point O "at ∞". That's the neutral (zero) element for $\boxplus$.

How does one define $P \boxplus P$? (Tangent line!)

Remarkably, the "addition" $P \boxplus Q$ is associative. (This is not obvious from the definition.)

Using $\boxplus$, we can construct new points: for instance, $(0, 3) \boxplus (1, -3) = (35, 207)$ as we will verify in the next example using Sage.

Easier to verify (but not producing anything new) is $(0, 3) \boxplus (1, 3) = (-1, -3)$.

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