1 Preparing for Midterm 1

- These problems are taken from the lectures to help you prepare for our upcoming midterm exam. You can find solutions to all of these in the lecture sketches.
- Additional, more exam-like, practice problems are also posted to our course website.



Example 1. Determine $4^{-1} \pmod{13}$.

Example 2. Solve $4x \equiv 5 \pmod{13}$.

(Bézout's identity) Let $a, b \in \mathbb{Z}$ (not both zero). There exist $x, y \in \mathbb{Z}$ such that

The integers x, y can be found using the **extended Euclidean algorithm**.

In particular, if $\gcd(a,b) = 1$, then $a^{-1} \equiv$

Example 3. Determine $16^{-1} \pmod{25}$.

Example 4. Determine $17^{-1} \pmod{23}$.

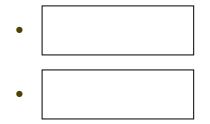
Definition 5. Euler's phi function $\phi(n)$ counts

If the prime factorization of n is $n = p_1^{k_1} \cdots p_r^{k_r}$, then $\phi(n) =$.

Example 6. Compute $\phi(35)$.

Example 7. Compute $\phi(100)$.

Our ultimate goal will be to secure messaging (at least) against:



Example 8. (affine cipher) A slight upgrade to the shift cipher, we encrypt each character as

 $E_{(a,b)}$: $x \mapsto ax + b \pmod{26}$.

How does the decryption work? How large is the key space?

Example 9. Encrypt *HOLIDAY* using a Vigenere cipher with key *BAD*.

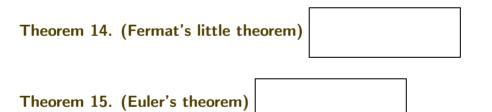
Example 10. In a few words, describe the following common kinds of attacks:

- ciphertext only attack
- known plaintext attack
- chosen plaintext attack
- chosen ciphertext attack

Example 11. Alice sends the ciphertext B K N D K G B Q to Bob. Somehow, Eve has learned that Alice is using the Vigenere cipher and that the plaintext is A L L C L E A R. Next day, Alice sends the message D N F F Q G E. Crack it and figure out the key that Alice used! (What kind of attack is this?)

Example 12. (substitution cipher) In a substitution cipher, the key k is some permutation of the letters A, B, ..., Z. For instance, k = FRA.... Then we encrypt $A \to F$, $B \to R$, $C \to A$ and so on. How large is the key space?

Example 13. It seems convenient to add the space as a 27th letter in the historic encryption schemes. Can you think of a reason against doing that?



- **Example 16.** Compute $3^{1003} \pmod{101}$.
- **Example 17.** Compute $3^{25} \pmod{101}$.
- **Example 18.** What are the last two (decimal) digits of 3^{7082} ?
- **Example 19.** Compute $2^{20} \pmod{41}$.
- **Example 20.** Express 25 in base 2.
- **Example 21.** Express 49 in base 2.
- **Example 22.** What is $(31)_8$ in decimal?

Example 23. What is $(FACE)_{16}$ in decimal?

Example 24. What is ASCII?

Example 25. Compute: $1011 \oplus 1111$

A **one-time pad** works as follows:

Example 26. Using a one-time pad with key k = 1100,0011, what is the message m = 1010, 1010 encrypted to?

If a one-time pad is used exactly once to encrypt a message, then perfect achieved.

is

Example 27. Alice made a mistake and encrypted the two plaintexts m_1 , m_2 using the same key k. How can Eve exploit that?

Using the one-time pad presents several challenges, including:



Example 28. Explain why a ciphertext only attack on the one-time pad is entirely hopeless. What about the other attacks?

Yet, the one-time pad by itself provides little protection of

Example 29. Alice sends an email to Bob using a one-time pad. Eve knows that and concludes that, per email standard, the plaintext must begin with To: Bob. Eve wants to tamper with the message and change it to To: Boo, for a light scare. Explain how Eve can do that!

Example 30. One thing that makes the one-time pad difficult to use is that the key needs to be the same length as the plaintext. What if we have a shorter key and just repeat it until it has the length we need? Why is that a terrible idea?

A stream cipher works as follows:

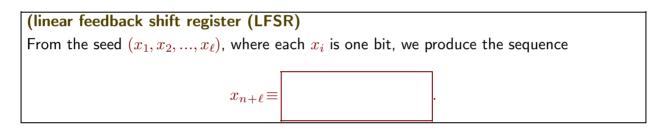
(linear congruential generator)	
From the seed x_0 , we produce the sequence x_{n+1} =	

Example 31. Generate values using the linear congruential generator $x_{n+1} = 5x_n + 3 \pmod{8}$, starting with the seed $x_0 = 6$. What is the period?

Example 32. Explain the idea behind using a **nonce** in a stream cipher.

Example 33. Let's use the PRG $x_{n+1} = 5x_n + 3 \pmod{8}$ as a stream cipher with the key $k = 4 = (100)_2$. The key is used as the seed x_0 and the keystream is $PRG(k) = x_1 x_2 \dots$ (where each x_i is 3 bits). Encrypt the message $m = (101 \ 111 \ 001)_2$.

Example 34. Eve intercepts the ciphertext $c = (111 \ 111 \ 111)_2$. It is known that a stream cipher with PRG $x_{n+1} = 5x_n + 3 \pmod{8}$ was used for encryption. Eve also knows that the plaintext begins with $m = (110 \ 1...)_2$. Help her crack the ciphertext!



Example 35. Which sequence is generated by the LFSR $x_{n+2} \equiv x_{n+1} + x_n \pmod{2}$, starting with the seed $(x_1, x_2) = (0, 1)$? What is the period?

Example 36. Which sequence is generated by the LFSR $x_{n+3} \equiv x_{n+1} + x_n \pmod{2}$, starting with the seed $(x_1, x_2, x_3) = (0, 0, 1)$? What is the period?

Example 37. Eve intercepts the ciphertext $c = (1111 \ 1011 \ 0000)_2$ from Alice to Bob. She knows that the plaintext begins with $m = (1100 \ 0...)_2$. Eve thinks a stream cipher using a LFSR with $x_{n+3} \equiv x_{n+2} + x_n \pmod{2}$ was used. If that's the case, what is the plaintext?

Example 38. One can also consider nonlinear recurrences (it mitigates some issues). Use $x_{n+3} \equiv x_{n+2}x_n + x_{n+1} \pmod{2}$ to generate some numbers.

A PRG is **predictable** if

Example 39. Let us consider a baby version of CSS. Our PRG uses the LFSR $x_{n+3} \equiv x_{n+1} + x_n \pmod{2}$ as well as the LFSR $x_{n+4} \equiv x_{n+2} + x_n \pmod{2}$. The output of the PRG is the output of these two LFSRs added with carry.

If we use (0,0,1) as the seed for LFSR-1, and (0,1,0,1) for LFSR-2, what are the first 10 bits output by our PRG?

Example 40. In each case, determine if the stream could have been produced by the LFSR $x_{n+5} \equiv x_{n+2} + x_n \pmod{2}$. If yes, predict the next three terms.

 $(STREAM-1) \dots, 1, 0, 0, 1, 1, 1, 1, 0, 1, \dots$ $(STREAM-2) \dots, 1, 1, 0, 0, 0, 1, 1, 0, 1, \dots$

Theorem 41. (Chinese Remainder Theorem)

Example 42. Solve $x \equiv 2 \pmod{5}$, $x \equiv 4 \pmod{7}$.

Example 43. Solve $x \equiv 1 \pmod{4}$, $x \equiv 2 \pmod{5}$, $x \equiv 3 \pmod{7}$.

Example 44.

- (a) Solve $x \equiv 2 \pmod{4}$, $x \equiv 3 \pmod{25}$.
- (b) Solve $x \equiv -1 \pmod{4}$, $x \equiv 2 \pmod{7}$, $x \equiv 0 \pmod{9}$.

Example 45.

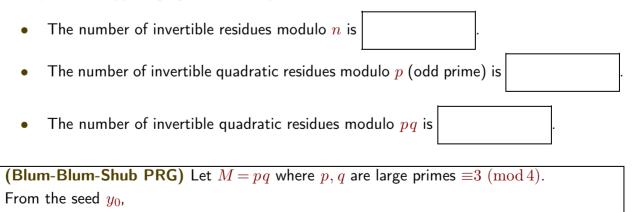
- (a) Let p > 3 be a prime. Show that $x^2 \equiv 9 \pmod{p}$ has exactly two solutions (i.e. ± 3).
- (b) Let p, q > 3 be distinct primes. Show that $x^2 \equiv 9 \pmod{pq}$ alwas has exactly four solutions (± 3 and two more solutions $\pm a$).

Example 46. Determine all solutions to $x^2 \equiv 9 \pmod{35}$.

Example 47. List all quadratic residues modulo 11.

Example 48. List all quadratic residues modulo 15. How many invertible quadratic residues are there? Explain!

Example 49. Suppose p, q are distinct primes.



Example 50. Generate random bits using the B-B-S PRG with M = 77 and seed 3.

Theorem 51.	-1 is a	quadratic	residue	modulo	(an	odd	prime)	p	\Leftrightarrow
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Fermat primality test Input: Output: Algorithm:

Example 52. If n is composite, then a is called a **Fermat liar** if