Midterm #2

MATH 227 — Calculus III
Friday, Apr 1

Please print your name:

Problem 1. Consider the function $f(x, y) = xy\cos(x + y)$. Determine the following:

(a) $f_x =$

(b) $f_{xy} =$

(c) $\nabla f =$

(d) The linearization of $f(x, y)$ at the point $(1, -1)$.

Solution.

(a) $f_x = y\cos(x + y) - xy\sin(x + y)$

(b) $f_{xy} = \cos(x + y) - y\sin(x + y) - x\sin(x + y) - xy\cos(x + y)$

(c) $\nabla f = \begin{bmatrix} y\cos(x + y) - xy\sin(x + y) \\ x\cos(x + y) - xy\sin(x + y) \end{bmatrix}$

(d) $L(x, y) = -1 + (-1)(x - 1) + 1(y + 1) = 1 - x + y$

(e) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ □

(e) In which direction does $f(x, y)$ at $(1, -1)$ increase most rapidly?

Problem 2. Write down a chain rule for $\frac{\partial}{\partial \theta} f$ for $f(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$.

Solution. $\frac{\partial}{\partial \theta} f = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -f_x r\sin\theta + f_y r\cos\theta$ □

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Problem 3. Consider the function $f(x, y, z) = 2 + x^2 - yz$.

(a) Find the derivative of $f(x, y, z)$ at $(1, -1, 2)$ in the direction $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

(b) Find an equation for the plane tangent to the surface $f(x, y, z) = 5$ at the point $(1, -1, 2)$.

Solution.

(a) The derivative of $f(x, y, z)$ at $(1, -1, 2)$ in direction $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ is

$$\nabla f\Big|_{(1,-1,2)} \cdot \frac{\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}}{\|\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}\|} = \begin{bmatrix} 2x \\ -z \\ -y \end{bmatrix}_{(1,-1,2)} \cdot \frac{1}{\sqrt{1+1+1}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{2-2-1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}.$$

(b) $\nabla f\Big|_{(1,-1,2)} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is a normal vector for the tangent plane, which is therefore of the form $2x - 2y + z = d$, and we find $d = 2 + 2 + 2 = 6$ using the point $(1, -1, 2)$. The tangent plane is $2x - 2y + z = 6$. ☐

Problem 4. Find all local extreme values and saddle points of the function $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution. To find the critical points, we need to solve the two equations $f_x = -6x + 6y = 0$ and $f_y = 6y - 6y^2 + 6x = 0$ for the two unknowns x, y.

[A general strategy is to solve one equation for one variable (in terms of the other), and substitute that in the other equation. Then we have a single equation in a single variable, which we can solve.]

Here, the first equation simplifies to $x = y$. Substituting that in the second equation, we get $6y - 6y^2 + 6y = 12y - 6y^2 = 6y(2 - y) = 0$. Hence, $y = 0$ or $y = 2$.

If $y = 0$ then $x = y = 0$, and we get the point $(0, 0)$. If $y = 2$ then $x = y = 2$, and we get the point $(2, 2)$.

In conclusion, the critical points are $(0, 0)$, $(2, 2)$.

$\left[f_{xx}f_{yy} - f_{xy}^2\right]_{(0,0)} = \left[(-6)\cdot(6 - 12y) - 6^2\right]_{(0,0)} = -72 < 0$. Hence, $(0, 0)$ is a saddle point.

$\left[f_{xx}f_{yy} - f_{xy}^2\right]_{(2,2)} = \left[(-6)\cdot(6 - 12y) - 6^2\right]_{(2,2)} = 72 > 0$ and $f_{xx} = -6 < 0$. Hence, $(2, 2)$ is a local max. ☐

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Problem 5. Consider the integral $\int_0^2 \int_0^{x^2} (1+2xy)\mathrm{d}y\mathrm{d}x$.

(a) Evaluate the integral.

(b) Interchange the order of integration. Do not evaluate this second integral.

Solution.

(a) $\int_0^2 \int_0^{x^2} (1+2xy)\mathrm{d}y\mathrm{d}x = \int_0^2 \left[y+xy^2\right]_{y=0}^{y=x^2} \mathrm{d}x = \int_0^2 (x^2+x^5)\mathrm{d}x = \left[\frac{x^3}{3}+\frac{x^6}{6}\right]_{x=0}^{x=2} = \frac{8}{3}+\frac{32}{3} = \frac{40}{3}$

(b) Make a sketch! The range for y is $0 \leqslant y \leqslant 4$. The horizontal cross-sections corresponding to y are described by $\sqrt{y} \leqslant x \leqslant 2$.

Hence, $\int_0^2 \int_0^{x^2} (1+2xy)\mathrm{d}y\mathrm{d}x = \int_0^4 \int_{\sqrt{y}}^2 (1+2xy)\mathrm{d}x\mathrm{d}y$.

For comparison:

$$\int_0^4 \int_{\sqrt{y}}^2 (1+2xy)\mathrm{d}x\mathrm{d}y = \int_0^4 \left[x+x^2y\right]_{x=\sqrt{y}}^{x=2} \mathrm{d}x = \int_0^4 (2+4y-\sqrt{y}-y^2)\mathrm{d}y = \left[2y+2y^2-\frac{2y^{3/2}}{3}-\frac{y^3}{3}\right]_0^4 = \frac{40}{3}$$

□

Problem 6. Convert the cartesian integral $\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{1}{1+x^2+y^2}\mathrm{d}y\mathrm{d}x$ into an equivalent polar integral.

Do not evaluate either of these integrals.

Solution. $\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{1}{1+x^2+y^2}\mathrm{d}y\mathrm{d}x = \int_0^{\pi/2} \int_0^2 \frac{r}{1+r^2}\mathrm{d}r\mathrm{d}\theta$

[On the other hand, $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \frac{1}{1+x^2+y^2}\mathrm{d}y\mathrm{d}x = \int_0^{\pi} \int_0^3 \frac{r}{1+r^2}\mathrm{d}r\mathrm{d}\theta$.] □

Problem 7. Determine a system of equations for finding the extreme values of $f(x,y,z) = x-y+2z$ on the sphere $x^2+y^2+z^2=3$. Do not attempt to solve this system of equations.

Solution. Let $g(x,y,z) = x^2+y^2+z^2$. By the method of Lagrange multipliers, we need to find values x,y,z,λ such that

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x,y,z) = 3.$$

Since $\nabla f = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$, these equations become:

$$\begin{aligned} 1 &= 2\lambda x \\ -1 &= 2\lambda y \\ 2 &= 2\lambda z \\ x^2+y^2+z^2 &= 3 \end{aligned}$$

These are four equations and four unknowns. Solving this system, we expect to get a handful of individual solutions. These then are the candidates for local extrema. □