Preparing for Midterm #2

MATH 126 — Calculus III
Friday, Apr 1

Please print your name:

Problem 1. Consider the function $f(x, y) = \frac{1}{1 + 2x^2 - xy}$.

(a) What is the natural domain of $f(x, y)$?

(b) Compute the partial derivatives f_x and f_{xy}.

(c) Find the linearization of $f(x, y)$ at $(2, 3)$.

(d) Compute the gradient ∇f.

(e) Show that $f(x, y)$ is a solution to the partial differential equation $x\frac{\partial f}{\partial x} + (4x - y)\frac{\partial f}{\partial y} = 0$.

(f) Determine and sketch the level curve $f(x, y) = 1$.

(g) Find a vector which is orthogonal to the curve $f(x, y) = 1$ at the point $(1, 2)$.

[Make sure to compare your answer to what you got for the level curve $f(x, y) = 1$.]

(h) Find the derivative of $f(x, y)$ at $(1, 2)$ in direction $\boldsymbol{v} = 3\boldsymbol{i} + \boldsymbol{j}$.

(i) Find a vector which is orthogonal to the curve $f(x, y) = 2$ at the point $(1/2, 2)$.

(j) In which direction does $f(x, y)$ at $(1/2, 2)$ increase most rapidly?

(k) Find the equation for the plane tangent to the graph of $f(x, y)$ at $(1, 2)$.

(l) Let $w = f(x, y)$ and $x = 2 + t$, $y = \cos(t)$. Find $\frac{\mathrm{d}w}{\mathrm{d}t}$ (in terms of t) in two ways:

- by expressing w in terms of t and differentiating directly,
- by using the chain rule.

(m) Find the local extreme values and saddles of $f(x, y)$.

Problem 2.

(a) Find all local extreme values and saddle points of the function $f(x, y) = \ln(x + y) + x^2 - y$.

(b) Find all local extreme values and saddle points of the function $f(x, y) = x + 2x^2 + x^3 + xy + y^2$.

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Problem 3.

(a) Let $g(x)$ be a function of one variable such that $g'(x) = e^{x^2}$. Let $w = g(st + e^t)$. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

(b) Let $f(x, y)$ be some function of two variables. Write down a chain rule for $\frac{\partial}{\partial w} f(x(u, v, w), y(u, v, w))$.

(c) Write down a chain rule for $\frac{\partial}{\partial r} f$ and $\frac{\partial}{\partial \theta} f$ for $f(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$.

(d) **(Challenge!)** Write down a chain rule for $\frac{\partial^2}{\partial r^2} f$ for $f(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$.

Problem 4. Consider the function $f(x, y, z) = xyz^2 + 4\sqrt{3 + yz}$.

(a) Compute the gradient ∇f.

(b) Find the linearization of $f(x, y, z)$ at $(2, 1, 1)$.

(c) Find the derivative of $f(x, y, z)$ at $(2, 1, 1)$ in direction $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

(d) Compute the partial derivative f_{zyx}.

(e) Determine a normal vector for the surface $f(x, y, z) = 7$ at $(-1, 1, 1)$.

(f) Find equations for the tangent plane and normal line for the surface $f(x, y, z) = 7$ at the point $(-1, 1, 1)$.

(g) Find the line tangent to the curve of intersection of the surfaces $x^2yz = 1$ and $f(x, y, z) = 7$ at the point $(-1, 1, 1)$.

Problem 5.

(a) Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x^2 - z^2 - 1 = 0$.

In other words, find the point(s) on (the hyperbolic cylinder) $x^2 - z^2 - 1 = 0$ that are closest to the origin.

(b) Determine a system of equations for finding the extreme values of $f(x, y, z) = x - y + 2z$ on the sphere $x^2 + y^2 + z^2 = 3$.

In this case, it is actually not hard to solve that system. You will find two candidates for extrema. For geometric reasons, one of these has to be a maximum and the other a minimum. (Can you explain why that has to be the case?)

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Problem 6. Consider the iterated integral $\int_0^4 \int_{2-x/2}^{\sqrt{4-x}} xy\,\mathrm{d}y\,\mathrm{d}x$.

(a) Evaluate the integral.

(b) Interchange the order of integration.

(If you have time, evaluate that second integral and verify that it gives the same value.)

Problem 7. Consider the region R with $x^2+y^2 \leqslant 4$ and $y \geqslant 0$. Write down an iterated integral for the area of R

(a) using vertical cross-sections,

(b) using horizontal cross-sections,

(c) using polar coordinates.

Problem 8. Convert the cartesian integral $\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{1}{1+x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$ into an equivalent polar integral. Then evaluate the polar integral.