Example 130. Find the line integral of $f(x,y,z)=x+z$ over the straight-line segment from $(1,1,0)$ to $(3,2,2)$.

Solution. We use the parametrization $\boldsymbol{r}(t)=\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} 1\\ 1\\ 0 \end{bmatrix}+t\begin{bmatrix} 2\\ 1\\ 2 \end{bmatrix}$, $t\in[0,1]$. Then, $\|\boldsymbol{r}'(t)\|=3$.

$$\int_C (x+z)\mathrm{d}s=\int_0^1 ((1+2t)+2t)3\mathrm{d}t=3\Big[t+2t^2\Big]_0^1=9.$$

Example 131. Find the average value of $f(x,y)=y$ on the upper half of the unit circle (only the circle $x^2+y^2=1$ itself, not any of its interior $x^2+y^2<1$).

Solution. We use the parametrization $\boldsymbol{r}(t)=\begin{bmatrix} x\\ y \end{bmatrix}=\begin{bmatrix} \cos(t)\\ \sin(t) \end{bmatrix}$, $t\in[0,\pi]$. Then, $\|\boldsymbol{r}'(t)\|=\left\|\begin{bmatrix} -\sin(t)\\ \cos(t) \end{bmatrix}\right\|=1$.

$$\int_C f(x,y)\mathrm{d}s=\int_0^\pi f(\cos(t),\sin(t))\cdot\|\boldsymbol{r}'(t)\|\mathrm{d}t=\int_0^\pi \sin(t)\mathrm{d}t=2$$

$$\text{avg}=\frac{1}{\text{length}(C)}\int_C f(x,y)\mathrm{d}s=\frac{2}{\pi}\approx 0.637$$

[Do you agree that the answer had to be $>\frac{1}{2}$?]

Exercise. For contrast, find the average value of $f(x,y)=y$ on the upper half of the unit disk (that is, the circle plus its interior $x^2+y^2\leqslant 1$). [Again, can you see that the answer had to be $<\frac{1}{2}$?]

This average is $\frac{1}{\pi/2}\int_0^\pi\int_0^1 r\sin(\theta)\,r\mathrm{d}r\mathrm{d}\theta=\frac{2}{\pi}\cdot\frac{1}{3}\cdot(1+1)=\frac{4}{3\pi}\approx 0.424$.

Line integrals (with respect to $\mathrm{d}x,\mathrm{d}y,\ldots$): $\int_C f(x,y,z)\mathrm{d}y=\int_a^b f(x(t),y(t),z(t))\,y'(t)\mathrm{d}t$

Since $\mathrm{d}y$ can be positive or negative, it matters (see next example) in which direction we are traversing the curve C (but, like in the case of line integrals with respect to $\mathrm{d}s$, the particular choice of parametrization for C does not matter).

Example 132. Compute $\int_C y\mathrm{d}y$, where C is the upper half of the unit circle $x^2+y^2=1$ from $(1,0)$ to $(-1,0)$.

Solution. We reuse the parametrization $\boldsymbol{r}(t)=\begin{bmatrix} x\\ y \end{bmatrix}=\begin{bmatrix} \cos(t)\\ \sin(t) \end{bmatrix}$, $t\in[0,\pi]$.

$$\int_C f(x,y)\mathrm{d}y=\int_0^\pi f(\cos(t),\sin(t))\cdot y'(t)\mathrm{d}t=\int_0^\pi \sin(t)\cos(t)\mathrm{d}t=\ldots=0.$$

Exercise. Evaluate the final integral in three different ways (substitution; trig identity; integration by parts).

Comment. If C' is the upper half of the unit circle $x^2+y^2=1$ from $(-1,0)$ to $(1,0)$ (that is, the same curve but traversed in the opposite direction), then $\int_{C'} f(x,y)\mathrm{d}y=-\int_C f(x,y)\mathrm{d}y$. Of course, in the present example, both integrals are just 0.

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