Triple integrals

The **volume** of a region D in 3D is given by $\iiint_D \mathrm{d}V = \iiint_D \mathrm{d}z\mathrm{d}y\mathrm{d}x$.

Example 118. Find the volume of the tetrahedron cut from the first octant by the plane $x+2y+z=2$.

First. Make a sketch of the tetrahedron! What are the intercepts? The triangle formed by these three points is one of the four sides of the tetrahedron.

Solution. (basic geometry) This is a simple enough object, that we can compute its volume directly: $\mathrm{vol} = \frac{1}{3}\cdot\mathrm{height}\cdot\mathrm{base} = \frac{1}{3}\cdot\left(\frac{1}{2}\cdot 2\cdot 1\right)\cdot 2 = \frac{2}{3}$ (here, we used as base the triangle in the xy-plane)

Solution. Let us set up an integral of the shape $\iiint_D \mathrm{d}z\mathrm{d}y\mathrm{d}x$.

- As a first step, realize that our object is completely described by the four inequalities

  $$x+2y+z\leqslant 2, \quad x\geqslant 0, \quad y\geqslant 0, \quad z\geqslant 0.$$

  Make sure you see how each inequality corresponds to one face of the tetrahedron!
- The overall range for x is $0\leqslant x\leqslant 2$.
  - This is obvious from the sketch.
  - On the other hand, just working from the inequalities, the only inequality of the form $x\leqslant\ldots$ is the first one: $x\leqslant 2-2y-z$. We don't know what y and z are, so we need to choose them so that $2-2y-z$ is as large as possible. Since $y\geqslant 0$ and $z\geqslant 0$, we get $x\leqslant 2-2\cdot 0-0=2$.
- With x specified, we now think about the corresponding range for y.
  - Clearly, $y\geqslant 0$.
  - The only inequality of the form $y\leqslant\ldots$ is the first one: $y\leqslant 1-\frac{x}{2}-\frac{z}{2}$. We don't know what z is, so we need to choose it so that $1-\frac{x}{2}-\frac{z}{2}$ is as large as possible. Since $z\geqslant 0$, we get $y\leqslant 1-\frac{x}{2}-\frac{0}{2}=1-\frac{x}{2}$.
- Finally, with x and y specified, we think about the corresponding range for z.
  - Clearly, $z\geqslant 0$.
  - The only inequality of the form $z\leqslant\ldots$ is the first one: $z\leqslant 2-x-2y$.
- Summarizing, we have described our tetrahedron by $0\leqslant x\leqslant 2$, $0\leqslant y\leqslant 1-\frac{x}{2}$, $0\leqslant z\leqslant 2-x-2y$.
- The corresponding integral is

$$\begin{aligned}\int_0^2\int_0^{1-x/2}\int_0^{2-x-2y}\mathrm{d}z\mathrm{d}y\mathrm{d}x &= \int_0^2\int_0^{1-x/2}(2-x-2y)\mathrm{d}y\mathrm{d}x\\ &= \int_0^2\left[2y-xy-y^2\right]_{y=0}^{y=1-x/2}\mathrm{d}x\\ &= \int_0^2\left(1-x+\frac{x^2}{4}\right)\mathrm{d}x = \frac{2}{3}\end{aligned}$$

Important comment. There are a total of $3!=6$ orderings of x,y,z in which we could have approached the problem. As an exercise, produce an integral of the form $\iiint_D \mathrm{d}y\mathrm{d}z\mathrm{d}x$ and check that its value is also $\frac{2}{3}$. The bounds you get in that case are: $0\leqslant y\leqslant 1$, $0\leqslant x\leqslant 2-2y$, $0\leqslant z\leqslant 2-x-2y$.

Armin Straub
straub@southalabama.edu