Example 116. Consider the integral $\int_0^1 \int_1^{e^x} \mathrm{d}y\mathrm{d}x$.

Sketch the region of integration and write an equivalent double integral with the order of integration reversed.

Solution. The range for y is $1 \leqslant y \leqslant e$. The horizontal cross-sections corresponding to y are described by $\ln(y) \leqslant x \leqslant 1$.

We thus obtain the equivalent double integral $\int_1^e \int_{\ln(y)}^1 \mathrm{d}x\mathrm{d}y$.

Example 117. Consider the region R described by $0 \leqslant x \leqslant 1$, $x \leqslant y \leqslant 1$.

(a) Write down an iterated integral for the area.

(b) Write down an iterated integral in polar coordinates for the area.

Solution.

(a) $\int_0^1 \int_x^1 \mathrm{d}y\mathrm{d}x = \frac{1}{2}$

(b) Make a sketch! Clearly, $\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2}$. Now, consider the ray with angle θ (to the x-axis) and think about the corresponding range for r. Basic trigonometry then shows that $0 \leqslant r \leqslant \csc\theta$.

$\int_{\pi/4}^{\pi/2} \int_0^{\csc\theta} r\mathrm{d}r\mathrm{d}\theta = \ldots = \frac{1}{2}$ [If you want to compute the integral, $\frac{\mathrm{d}}{\mathrm{d}\theta}\cot\theta = -\csc^2\theta$ is helpful.]

A quick summary of what we learned since last the midterm

We started working with functions $f(x, y)$, $g(x, y, z)$ of several variables:

- partial derivatives
- linearization
- chain rule
- gradient
  - directional derivative
  - direction of steepest descent
  - orthogonal to level curves/surfaces (tangent planes, ...)
- local extrema and saddle points
  - $\nabla f = 0$ and second derivative test
  - local extrema under constraints: Lagrange multipliers
- multiple integrals
  - interchange order of integration
  - polar coordinates (substitution)

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