Example 114. Consider the region $1 \leqslant r \leqslant 2$, $0 \leqslant \theta \leqslant \pi/2$ described in polar coordinates.

(a) Sketch the region.

(b) Write down a double integral for the area of the region using polar coordinates.

(c) Write down a double integral for the area using cartesian coordinates.

(d) Evaluate the polar integral and compare with the geometrically obvious answer.

Solution.

(a) This is the quarter of an annulus.

(b) In polar coordinates, the area is $\int_0^{\pi/2} \int_1^2 r \,\mathrm{d}r \mathrm{d}\theta$.

(c) Let us use vertical cross-sections (the region is symmetric, so it makes no difference) for cartesian coordinates. Clearly, the range for x is $0 \leqslant x \leqslant 2$.

For each x, we now need to describe the range of y in the corresponding cross-section. However, there is two cases that we need to distinguish here: for $1 \leqslant x \leqslant 2$ this range is $0 \leqslant y \leqslant \sqrt{4 - x^2}$, while for $0 \leqslant x \leqslant 1$ this range is $\sqrt{1 - x^2} \leqslant y \leqslant \sqrt{4 - x^2}$.

Putting these together, the area is $\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \mathrm{d}y \mathrm{d}x + \int_1^2 \int_0^{\sqrt{4-x^2}} \mathrm{d}y \mathrm{d}x$.

(d) Obviously, the area is $\frac{1}{4}[4\pi - \pi] = \frac{3\pi}{4}$. (Area of big disk minus area of small disk.)

$$\int_0^{\pi/2} \int_1^2 r \,\mathrm{d}r \mathrm{d}\theta = \frac{\pi}{2}\left[\frac{r^2}{2}\right]_{r=1}^{r=2} = \frac{3\pi}{4}$$

Example 115. Change the cartesian integral $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) \mathrm{d}y \mathrm{d}x$ into an equivalent polar integral.

Solution. As for any substitution, we need to take care of three things: the region of integration $\int_0^2 \int_0^{\sqrt{4-x^2}}$, the integrand $x^2 + y^2$ and the differentials $\mathrm{d}y \mathrm{d}x$.

- The region $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant \sqrt{4 - x^2}$ is the part of the disk of radius 2 which lies in the first quadrant. In polar coordinates, this region is described by $0 \leqslant \theta \leqslant \pi/2$, $0 \leqslant r \leqslant 2$.
- Using $x = r\cos\theta$, $y = r\sin\theta$, the integrand $x^2 + y^2$ gets replaced with $(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$. [Can you see directly, why $x^2 + y^2$ is r^2 in polar coordinates?]
- The differentials $\mathrm{d}y \mathrm{d}x$ get replaced with $r \mathrm{d}r \mathrm{d}\theta$. [In general, the Jacobian determinant J shows up here. Last time, we computed that for the substitution to polar coordinates that $J = r$.]

Putting these together, we get $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) \mathrm{d}y \mathrm{d}x = \int_0^{\pi/2} \int_0^2 r^3 \mathrm{d}r \mathrm{d}\theta$.

Nobody asked, but $\int_0^{\pi/2} \int_0^2 r^3 \mathrm{d}r \mathrm{d}\theta = \frac{\pi}{2}\left[\frac{r^4}{4}\right]_{r=0}^{r=2} = 2\pi$ is easy to compute (whereas the cartesian integral is much less pleasant to compute).

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