Example 111. (warmup) Compute the area of the region R bounded by $y=0$, $y=2x$, $x=1$.

Solution. (just a triangle!) This is a triangle with base 1 and height 2. Hence, its area is $\frac{1}{2}\cdot 1\cdot 2=1$.

Solution. (Calculus 2) We are talking about the area between the graphs of the functions $f(x)=2x$ and $g(x)=0$ for $x\in[0,1]$. This area is $\int_0^1 |f(x)-g(x)|\mathrm{d}x=\int_0^1 2x\,\mathrm{d}x=\left[x^2\right]_0^1=1$.

Solution. (Calculus 3, vertical cross-sections) $\int_0^1\int_0^{2x}\mathrm{d}y\mathrm{d}x=\int_0^1 2x\mathrm{d}x=\left[x^2\right]_0^1=1$

Solution. (Calculus 3, horizontal cross-sections) $\int_0^2\int_{y/2}^1\mathrm{d}x\mathrm{d}y=\int_0^2\left(1-\frac{y}{2}\right)\mathrm{d}y=\left[y-\frac{y^2}{4}\right]_0^2=1$

Substitution in multiple integrals

In $\int\int_R f(x,y)\mathrm{d}y\mathrm{d}x$, we want to make the change of variables $x=g(u,v)$, $y=h(u,v)$.

$$\int\int_R f(x,y)\mathrm{d}y\mathrm{d}x=\int\int_G f(g(u,v),h(u,v))|J(u,v)|\mathrm{d}v\mathrm{d}u$$

- where G is the region in the uv-plane corresponding to R,
- and $J(u,v)=\det\begin{bmatrix} g_u & g_v\\ h_u & h_v\end{bmatrix}=g_uh_v-g_vh_u$ is the **Jacobian determinant**.

Note that this is not suprising: when substituting a single variable $x=g(u)$ we need to substitute $\mathrm{d}x=g'(u)\mathrm{d}u$. The Jacobian determinant $J(u,\ v)$ (which, in a way, is the simplest combination of all involved partial derivatives) replaces the single derivative $g'(u)$.

Example 112. If $x=r\cos\theta$, $y=r\sin\theta$, then the Jacobian determinant is

$$J(r,\theta)=\det\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{bmatrix}=\det\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}=r\cos^2\theta+r\sin^2\theta=r.$$

Using **polar coordinates** $x=r\cos\theta$, $y=r\sin\theta$, we have

$$\int\int_R f(x,y)\mathrm{d}y\mathrm{d}x=\int\int_G f(r\cos\theta,r\sin\theta)r\mathrm{d}r\mathrm{d}\theta$$

Example 113. Determine $\int\int_R\mathrm{d}y\mathrm{d}x$ with the region R described by $x^2+y^2\leqslant 1$, $x\geqslant 0$, $y\geqslant 0$.

Solution. (just a circle!) Sketch the region! We are asked to find the area of a quarter of the unit circle. Obviously, the answer is going to be $\frac{\pi}{4}$.

Solution. (Cartesian coordinates, vertical cross-sections) $\int_0^1\int_0^{\sqrt{1-x^2}}\mathrm{d}y\mathrm{d}x=\int_0^1\sqrt{1-x^2}\mathrm{d}x=\ldots=\frac{\pi}{4}$.
However, the omitted steps do require some work (like a trigonometric substitution).

Solution. (polar coordinates) The region is described by $0\leqslant\theta\leqslant\frac{\pi}{2}$ and $0\leqslant r\leqslant r$.

Hence, in polar coordinates, $\int_0^{\pi/2}\int_0^1 r\mathrm{d}r\mathrm{d}\theta=\int_0^{\pi/2}\frac{1}{2}\mathrm{d}\theta=\frac{\pi}{4}$

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