Review. $\iint_R f(x,y)\mathrm{d}A$ where R is $0 \leqslant x \leqslant 2$, $x^2 \leqslant y \leqslant 2x$.

Taking (first vertical, then horizontal) cross-sections of R we arrive at the following two iterated integrals: $\int_0^2 \int_{x^2}^{2x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\mathrm{d}x\mathrm{d}y$.

The fact that we can interchange the order of integration is known as **Fubini's theorem**.

Example 109. $\iint_R \frac{\sin x}{x}\mathrm{d}y\mathrm{d}x$ where the region R is bounded by $y=0$, $y=x$, $x=1$.

Make a sketch of the region!

Solution. (vertical cross-sections) Here, we fix x and then let y range. The range for x is $0 \leqslant x \leqslant 1$. The corresponding range of y is $0 \leqslant y \leqslant x$ (from the point on $y=0$ to the point on $y=x$).

Hence, we get $\int_0^1 \int_0^x \frac{\sin x}{x}\mathrm{d}y\mathrm{d}x$. This is easy to compute! (Note that the integrand does not depend on y.)

$$\int_0^1 \int_0^x \frac{\sin x}{x}\mathrm{d}y\mathrm{d}x = \int_0^1 \frac{\sin x}{x}(x-0)\mathrm{d}x = \int_0^1 \sin x\,\mathrm{d}x = \Big[-\cos x\Big]_{x=0}^{x=1} = 1 - \cos 1.$$

Solution. (horizontal cross-sections) Here, we fix y and then let x range. The range for y is $0 \leqslant y \leqslant 1$. The corresponding range of x is $y \leqslant x \leqslant 1$ (from the point on $y=x$ to the point on $x=1$).

Hence, we get $\int_0^1 \int_y^1 \frac{\sin x}{x}\mathrm{d}x\mathrm{d}y$.

In contrast to the previous case, we get stuck with this integral. That's because it is not possible to write down an antiderivative for $\frac{\sin x}{x}$ using our repertoire of functions. (Of course, the integral has the same value as the one before.)

Note. This example illustrates that interchanging the order of integration can make a huge difference.

> The **area** of a region R in 2D is given by $\iint_R \mathrm{d}A$.

Just like $\int_a^b \mathrm{d}x = b-a$ is the **length** of the interval $[a,b]$, and $\iiint_D \mathrm{d}V$ is the **volume** of a region D in 3D.

The integral above is often taken as the definition of area!

Example 110. Consider the region R with $x^2+y^2 \leqslant 1$ and $x+y \geqslant 1$. Write down an iterated integral for the area of R using vertical cross-sections.

Solution. Make a sketch! (See Figure 14.14 in our book.)

Vertical cross-sections means fixing x (with $0 \leqslant x \leqslant 1$) and deciding on the appropriate range for y. The sketch reveals that this range is $1-x \leqslant y \leqslant \sqrt{1-y^2}$. Hence, $\mathrm{area}(R) = \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} \mathrm{d}y\mathrm{d}x$.

Exercise. Compute this iterated integral. (For geometric reasons, we already know that $\mathrm{area}(R) = \frac{\pi}{4} - \frac{1}{2}$.)

Note. Since x and y play a symmetric role in the definition of R, horizontal cross-sections will lead to the same integral (with x and y swapped). Do it!

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