Functions of several variables

Vocabulary. A subset $S \subset \mathbb{R}^2$ (likewise for $\mathbb{R}^3$) is **bounded** if there is some M such that $\|\boldsymbol{x}\| < M$ for all $\boldsymbol{x} \in S$. Otherwise, S is called **unbounded**.

$\boldsymbol{x} \in S$ is an **interior point** if S contains an entire [possibly small] disk with center $\boldsymbol{x}$.

[Equivalently, if there is some $\varepsilon > 0$ such that all $\boldsymbol{y} \in \mathbb{R}^2$ with $\|\boldsymbol{x} - \boldsymbol{y}\| < \varepsilon$ are also contained in S.]

$\boldsymbol{y} \in \mathbb{R}^2$ is a **boundary point** of S is every disk around $\boldsymbol{y}$ has points from S and not from S.

The set S is **open** if every $\boldsymbol{x} \in S$ is an interior point.

The set S is **closed** if it contains all its boundary points.

Example 74. $f(x, y) = \sqrt{2 - x - 2y}$

- Just checking: what is $f(1, -2)$?
- Find and sketch the natural domain of this function of two variables.

  Is it bounded or unbounded? Is it open? Is it closed? What is its boundary (i.e. boundary points)?
- Sketch the **level curves** $f(x, y) = c$ for $c = 0, 1, 2, 3$.

  [You have surely seen level curves (countour lines) on geographical maps, where they indicate elevation. See, for instance, Figure 13.7 in our book.]
- Plot $f(x, 0)$ and compare with the level curves.

Solution.

- $f(1, -2) = \sqrt{2 - 1 - 2(-2)} = \sqrt{5}$
- The natural domain consists of all points (x, y) such that $2 - x - 2y \geqslant 0$ (the only trouble with this function is that, since we work with real numbers, we shouldn't have something negative under $\sqrt{\ }$).

  Formally, the domain is $\{(x, y) \in \mathbb{R}^2 : 2 - x - 2y \geqslant 0\}$.

  To sketch it, first think about $2 - x - 2y = 0$. This is a line. The easiest way to plot it is to determine x-intercept and y-intercept: this is the line through $(2, 0)$ and $(0, 1)$. To find out, on which side of the line our domain lies, we can check any convenient point: for instance, $(0, 0)$ is in our domain, and so the domain consists of everything on and below the line.
  - Our domain is unbounded.
  - It is not open.
  - It is closed.
  - Its boundary consists of the line described by $2 - x - 2y = 0$.
- $c = 0$: $\sqrt{2 - x - 2y} = 0$ implies $2 - x - 2y = 0$. So, the level curve $f(x, y) = 0$ is the line $2 - x - 2y = 0$.

  $c = 1$: $\sqrt{2 - x - 2y} = 1$ implies $2 - x - 2y = 1$. So, the level curve $f(x, y) = 1$ is the line $2 - x - 2y = 1$.

  This line is parallel to $f(x, y) = 0$. [Why?! Also, think about what we learned about parallel planes.]

  $c = 2$: $\sqrt{2 - x - 2y} = 2$ implies $2 - x - 2y = 4$. So, the level curve $f(x, y) = 2$ is the line $2 - x - 2y = 4$.

  $c = 3$: $\sqrt{2 - x - 2y} = 3$ implies $2 - x - 2y = 9$. So, the level curve $f(x, y) = 3$ is the line $2 - x - 2y = 9$.

  Both of these lines are parallel to $f(x, y) = 0$ as well. The spacing between these lines increases.
- Do it! Make sure that the increase in spacing between the level curves makes sense from your plot.

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