Example 69. Let $P(t)=\begin{bmatrix}\cos(t)\\ \sin(t)\\ t\end{bmatrix}$ be the position of a particle at time t. Find its velocity, speed and acceleration at time t. What is the distance traveled between $t=0$ and $t=2\pi$?

Solution. The velocity is $\boldsymbol{v}(t)=P'(t)=\begin{bmatrix}-\sin(t)\\ \cos(t)\\ 1\end{bmatrix}$. The speed is $|\boldsymbol{v}(t)|=\sqrt{(-\sin t)^2+(\cos t)^2+1^2}=\sqrt{2}$. (Again, constant speed!)

The acceleration is $\boldsymbol{a}(t)=P''(t)=\begin{bmatrix}-\cos(t)\\ -\sin(t)\\ 0\end{bmatrix}$. (Because of constant speed, $\boldsymbol{a}\cdot\boldsymbol{v}=0$. Check!)

The **distance traveled** is $\int_0^{2\pi}|\boldsymbol{v}(t)|\,\mathrm{d}t=2\pi\sqrt{2}$.

[Recall from Lecture 3 that we just computed the **arc length** of the parametric curve $P(t)$ for $t\in[0,2\pi]$.]

The **acceleration** splits into two natural parts: $\boldsymbol{a}=\boldsymbol{a}_T+\boldsymbol{a}_N$

- The **tangential part** $\boldsymbol{a}_T$ of the acceleration is the projection of the acceleration in the direction of motion: $\boldsymbol{a}_T=\mathrm{proj}_{\boldsymbol{v}}\,\boldsymbol{a}=\frac{\boldsymbol{a}\cdot\boldsymbol{v}}{|\boldsymbol{v}|^2}\boldsymbol{v}$.
- The other part of the acceleration is the **normal part** $\boldsymbol{a}_N=\boldsymbol{a}-\boldsymbol{a}_T$.

Why is the normal part always perpendicular to the direction of motion? (This explains the "normal".)

$\boldsymbol{a}_T$ is the projection of $\boldsymbol{a}$ onto $\boldsymbol{v}$. The "error" $\boldsymbol{a}_N=\boldsymbol{a}-\boldsymbol{a}_T$ is always perpendicular to the projection.

Example 70. Let $P(t)=\begin{bmatrix}\cos(t^2)\\ \sin(t^2)\end{bmatrix}$ be the position of a particle at time t. Find its velocity, speed and acceleration at time t. Determine the tangential and normal part of the acceleration.

Note. This particle is still traveling along a circle. However, the particle is speeding up with increasing time.

Solution. The velocity is $\boldsymbol{v}(t)=P'(t)=\begin{bmatrix}-2t\sin(t^2)\\ 2t\cos(t^2)\end{bmatrix}$.

The speed is $|\boldsymbol{v}(t)|=\sqrt{(-2t\sin(t^2))^2+(2t\cos(t^2))^2}=2|t|$. (Our particle is speeding up with t.)

The acceleration is $\boldsymbol{a}(t)=P''(t)=\begin{bmatrix}-2\sin(t^2)-4t^2\cos(t^2)\\ 2\cos(t^2)-4t^2\sin(t^2)\end{bmatrix}$.

For the tangential part $\boldsymbol{a}_T$, we need to project $\boldsymbol{a}(t)$ onto $\boldsymbol{v}(t)$. This projection is (recall that $|\boldsymbol{v}|^2=4t^2$)

$$\frac{\boldsymbol{a}\cdot\boldsymbol{v}}{|\boldsymbol{v}|^2}\boldsymbol{v}=\frac{4t}{4t^2}\begin{bmatrix}-2t\sin(t^2)\\ 2t\cos(t^2)\end{bmatrix}=\begin{bmatrix}-2\sin(t^2)\\ 2\cos(t^2)\end{bmatrix},$$

where we used $|\boldsymbol{v}|^2=4t^2$ and

$$\boldsymbol{a}\cdot\boldsymbol{v}=-2t\sin(t^2)[-2\sin(t^2)-4t^2\cos(t^2)]+2t\cos(t^2)[2\cos(t^2)-4t^2\sin(t^2)]=4t.$$

The normal part of the acceleration then is $\boldsymbol{a}_N=\boldsymbol{a}-\boldsymbol{a}_T=\begin{bmatrix}-4t^2\cos(t^2)\\ -4t^2\sin(t^2)\end{bmatrix}$.

[In this particular case, we could have seen this decomposition of $\boldsymbol{a}$ by staring at it. Do you see it in hindsight?]

Remark 71. What we just did in 2D, can be done in any higher dimension.

In particular, we have two important (perpendicular!) directions associated with a curve:

- The **unit tangent vector** $\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ (unless $\boldsymbol{v}=\boldsymbol{0}$).
- The **(principal) unit normal vector** $\boldsymbol{N}=\frac{\boldsymbol{a}_N}{|\boldsymbol{a}_N|}$ (unless $\boldsymbol{a}_N=\boldsymbol{0}$).
- If in 3D, then the **unit binormal vector** $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}$ is a third perpendicular direction.

Together, $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$ are a natural substitute for $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ when working with the curve at hand. The corresponding coordinate system is often called the $\boldsymbol{TNB}$ frame (or, **Frenet frame**).