Example 52. Consider the plane $x+2y+z=2$.

(a) Find a normal vector for the plane.

(b) Find the intersections of the plane with each of the three coordinate axes.

(c) Sketch the plane.

Solution.

(a) $\boldsymbol{n}=\langle 1,2,1\rangle$

(b) To find the intersection with the x-axis, we set $y=0$ and $z=0$ in the equation for the plane: $x+2\cdot 0+0=2$, which we "solve" to get $x=2$. The point of intersection is $(2,0,0)$. The value 2 (or sometimes the point) is called the **x-intercept** of the plane.

Likewise, the y-intercept is 1 and the z-intercept is 2. The corresponding intersection points are $(0,1,0)$ and $(0,0,2)$.

(c) Mark the three point we just found in a coordinate system, and add lines through each of the three sides of the triangle to indicate our plane.

[Look at your sketch: can you see (at least roughly) how the normal vector is indeed perpendicular to the plane?]

Example 53. Find an equation for the plane parallel to $x+2y+z=2$ through $(2,3,7)$.

Solution. Since the planes are parallel, they have the same normal direction.

Our plane can therefore also be written as $x+2y+z=d$ for some d.

To find d, we use that $(x,y,z)=(2,3,7)$ is a point on the plane: $2+2\cdot 3+7=d$, so $d=15$.

Example 54. Is the line $x=1-t$, $y=2t$, $z=3-4t$ parallel to the plane $x+2y+z=2$?

Solution. The line has direction vector $\boldsymbol{v}=\langle -1,2,-4\rangle$.

[Note that it can be written as $(x,y,z)=(1-t,2t,3-4t)=\underbrace{(1,0,3)}_{\text{point}}+t\,\underbrace{\langle -1,2,-4\rangle}_{\text{direction}}$.]

The plane has normal vector $\boldsymbol{n}=\langle 1,2,1\rangle$.

The line is perpendicular to the plane if and only if $\boldsymbol{v}\cdot\boldsymbol{n}=0$. (Why?!) But $\boldsymbol{v}\cdot\boldsymbol{n}=-1+4-4=-1\neq 0$. Hence, the line is not parallel to the plane.

Example 55. (intersecting two planes) Consider the two planes

$$x+2y+z=2, \qquad x+y-2z=1.$$

These planes are not parallel (why?!) and so they intersect in a line. Determine that line.

Solution. There are different approaches. Here is one that gives you a taste of linear algebra:

- Thinking just in terms of equations, we have two equations but three "unknowns". This means that we cannot solve for all three variables; one of them needs to be otherwise specified. Take (for instance) z and specify $z=t$ where t is some parameter (this will be the parameter for our line).

  [In linear algebra, we would say that z is a "free variable".]
- Our equations now are $x+2y=2-t$ and $x+y=1+2t$.

  [It is not necessary but customary to move the nonvariable parts to the right-hand side.]
- These are two equations and two unknowns (x,y are the unknowns; t is some value), so we can solve for x and y. For instance, we can subtract the second equation from the first (with the intent to eliminate x): We get $y=(2-t)-(1+2t)=1-3t$.

  To find x, we substitute that in the first equation and get $x+2(1-3t)=2-t$, which gives $x=5t$.
- Taken together, we have $\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} 5t\\ 1-3t\\ t \end{bmatrix}$. This is a parametrization of the line.

  [Note that $\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} 5t\\ 1-3t\\ t \end{bmatrix}=\begin{bmatrix} 0\\ 1\\ 0 \end{bmatrix}+t\begin{bmatrix} 5\\ -3\\ 1 \end{bmatrix}$, so we have the usual "point + direction" parametrization.]

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