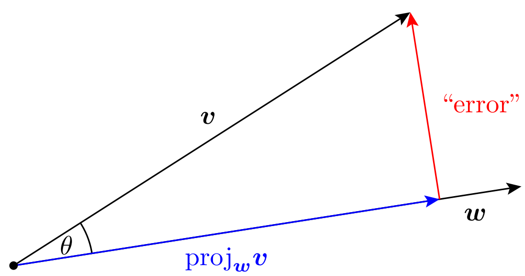
Review. $\boldsymbol{v}\cdot\boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3 = |\boldsymbol{v}|\,|\boldsymbol{w}|\cos\theta$ where $\theta \in [0,\pi]$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$

Projections

The **projection** of $\boldsymbol{v}$ onto $\boldsymbol{w}$ is the vector into the direction of $\boldsymbol{w}$, which is as close as possible to $\boldsymbol{v}$.

This projection is denoted by $\mathrm{proj}_{\boldsymbol{w}}\,\boldsymbol{v}$ and it is a vector (actually, a multiple of $\boldsymbol{w}$).

Note. From the sketch, we see that $\boldsymbol{v} = \mathrm{proj}_{\boldsymbol{w}}\,\boldsymbol{v} + $ "error" and that the error is orthogonal to $\boldsymbol{w}$.



Basic trigonometry tells us that the length of $\mathrm{proj}_{\boldsymbol{w}}\,\boldsymbol{v}$ is $|\boldsymbol{v}|\cos\theta$. Hence:

$$\mathrm{proj}_{\boldsymbol{w}}\,\boldsymbol{v} = \underbrace{|\boldsymbol{v}|\cos\theta}_{\text{length}} \quad \underbrace{\frac{\boldsymbol{w}}{|\boldsymbol{w}|}}_{\substack{\text{direction}\\ \text{(scaled to length 1)}}}$$

$$= \frac{|\boldsymbol{v}|\,|\boldsymbol{w}|\cos\theta}{|\boldsymbol{w}|}\,\frac{\boldsymbol{w}}{|\boldsymbol{w}|} = \left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{w}|^2}\right)\boldsymbol{w}$$

The **projection** of $\boldsymbol{v}$ onto $\boldsymbol{w}$ is $\mathrm{proj}_{\boldsymbol{w}}\,\boldsymbol{v} = \left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{w}|^2}\right)\boldsymbol{w}$.

In particular, if $\boldsymbol{w}$ is a unit vector then $\boxed{\mathrm{proj}_{\boldsymbol{w}}\,\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{w})\,\boldsymbol{w}}$.

Example 31. What is the projection of $\boldsymbol{v} = \langle 2,1\rangle$ onto $\boldsymbol{w} = \langle 3,0\rangle$?

Solution. Make a sketch! Make sure that it is obvious to you that $\mathrm{proj}_{\boldsymbol{w}}\,\boldsymbol{v} = \langle 2,0\rangle$.

Solution. $\mathrm{proj}_{\boldsymbol{w}}\,\boldsymbol{v} = \left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{w}|^2}\right)\boldsymbol{w} = \frac{6}{9}\langle 3,0\rangle = \langle 2,0\rangle$

Note. Writing $\boldsymbol{v} = \mathrm{proj}_{\boldsymbol{w}}\,\boldsymbol{v} + $ "error", we get $\langle 2,1\rangle = \langle 2,0\rangle + \langle 0,1\rangle$.

The "error" is orthogonal to $\boldsymbol{w}$: $\langle 0,1\rangle\cdot\langle 3,0\rangle = 0$. (This check guarantees that we projected correctly!)

Example 32. What is the projection of $\boldsymbol{v} = \langle 1,2,3\rangle$ onto $\boldsymbol{w} = \langle 1,1,1\rangle$?

Solution. $\mathrm{proj}_{\boldsymbol{w}}\,\boldsymbol{v} = \left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{w}|^2}\right)\boldsymbol{w} = \frac{6}{3}\langle 1,1,1\rangle = \langle 2,2,2\rangle$

Note. Writing $\boldsymbol{v} = \mathrm{proj}_{\boldsymbol{w}}\,\boldsymbol{v} + $ "error", we get $\langle 1,2,3\rangle = \langle 2,2,2\rangle + \underbrace{\langle -1,0,1\rangle}_{\langle 1,2,3\rangle - \langle 2,2,2\rangle}$.

The "error" is orthogonal to $\boldsymbol{w}$: $\langle -1,0,1\rangle\cdot\langle 1,1,1\rangle = 0$. (This check guarantees that we projected correctly!)

We will next discuss the cross product. For that, we need the notion of **right-handedness**:

- The coordinate system with x, y and z-axis, as we have used it so far, is right-handed:

  Take your right hand, and let the thumb point in x direction. Then form a "gun" with your thumb and index finger, and let the index finger point in y direction. When put at a right angle to the index finger, your middle finger points in z direction.

- There is nothing wrong with a left-handed coordinate system (for instance, the z axis could point "down" instead of "up"). However, there is a difference between the two: no rotating or moving around will turn a right-handed system into a left-handed one.

  This difference is usally referred to as **orientation** or **handedness**.

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