Example 25. Sketch two random vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in standard position. Then, also sketch:

(a) $\boldsymbol{v}+\boldsymbol{w}$

(b) $2\boldsymbol{w}$

(c) $-\boldsymbol{w}$

(d) $\boldsymbol{v}-\boldsymbol{w}$

The dot product

If $\boldsymbol{v}=\langle v_1, v_2, v_3\rangle$ and $\boldsymbol{w}=\langle w_1, w_2, w_3\rangle$, then their **dot product** is $\boldsymbol{v}\cdot\boldsymbol{w}=v_1w_1+v_2w_2+v_3w_3$. Likewise, if $\boldsymbol{v}=\langle v_1, v_2\rangle$ and $\boldsymbol{w}=\langle w_1, w_2\rangle$, then $\boldsymbol{v}\cdot\boldsymbol{w}=v_1w_1+v_2w_2$.

Note that the dot product of two vectors is a scalar! That's why the dot product is also called **scalar product**. (Another name you might hear it referred to is **inner product**.)

Example 26. Let $\boldsymbol{v}=2\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{w}=\boldsymbol{j}+3\boldsymbol{k}$. Compute the following:

(a) $\boldsymbol{v}\cdot\boldsymbol{w}$

(b) $\boldsymbol{w}\cdot\boldsymbol{v}$

(c) $\boldsymbol{v}\cdot\boldsymbol{v}$

(d) $\boldsymbol{v}\cdot(\boldsymbol{v}+\boldsymbol{w})$

Solution.

(a) $\boldsymbol{v}\cdot\boldsymbol{w}=\langle 2,-1,1\rangle\cdot\langle 0,1,3\rangle=2\cdot 0+(-1)\cdot 1+1\cdot 3=2$

(b) $\boldsymbol{w}\cdot\boldsymbol{v}=\ldots=2$ as well. It is clear that always $\boldsymbol{v}\cdot\boldsymbol{w}=\boldsymbol{w}\cdot\boldsymbol{v}$. In other words, the dot product is commutative.

(c) $\boldsymbol{v}\cdot\boldsymbol{v}=2^2+(-1)^2+1^2=6=|\boldsymbol{v}|^2$. Again, it is clear that always $\boxed{\boldsymbol{v}\cdot\boldsymbol{v}=|\boldsymbol{v}|^2}$.

(d) $\boldsymbol{v}\cdot(\boldsymbol{v}+\boldsymbol{w})=\langle 2,-1,1\rangle\cdot\langle 2,0,4\rangle=4+0+4=8$

This is the same as $\boldsymbol{v}\cdot(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{w}=6+2=8$.

We always have $\boldsymbol{v}\cdot(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{w}$. In other words, the dot product is distributive.

Example 27. Let $\boldsymbol{v}=\langle -1,1\rangle$, $\boldsymbol{w}=\langle 2,2\rangle$. Then $\boldsymbol{v}\cdot\boldsymbol{w}=0$. What is the geometric significance?

- Make a sketch!
- Realize that the two vectors are orthogonal (in other words, perpendicular, or at 90° angle).

You might also notice that $\boldsymbol{w}$ is precisely 2 times as long as $\boldsymbol{v}$. That's an interesting observation, but does not explain why $\boldsymbol{v}\cdot\boldsymbol{w}=0$. [For instance, for $\boldsymbol{v}=\langle -1,1\rangle$ and $\boldsymbol{w}=\langle 3,3\rangle$, we still have $\boldsymbol{v}\cdot\boldsymbol{w}=0$.]

Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **orthogonal** if and only if $\boldsymbol{v}\cdot\boldsymbol{w}=0$.

Appreciate how surprisingly simple it is to decide whether two vectors are at a right angle!
Of course, behind all this is... Pythagoras! More next time.

Armin Straub
straub@southalabama.edu