Review. Cartesian coordinates and points in three dimensions; distances

Example 18. Interpret the following equations (in three-dimensional space) geometrically:

(a) $z = 2$

(b) $x^2 + y^2 = 1$

(c) $x^2 + y^2 = 1$, $z = 2$

(d) $x^2 + y^2 + z^2 = 4$

Solution.

(a) This is a plane parallel to the xy-plane.
Equivalently, we could describe this as a plane perpendicular to the z-axis.
Comment. This is a 2-dimensional object. (1 equation in 3D: $3 - 1 = 2$)

(b) [In the xy-plane, this is just the unit circle.]
In three dimensions, this is (the surface) of a cylinder (or tube of infinite height). Make a sketch!
Comment. Again, this is a 2-dimensional object. (1 equation: $3 - 1 = 2$)

(c) Geometrically, this is the intersection of our previous two objects. What is left, is a circle in the plane $z = 2$.
Comment. Now, this is a 1-dimensional object. (2 equations: $3 - 2 = 1$)

(d) By our knowledge of distance, we see that these are all the points (x, y, z) that have distance 2 from the origin. This is called a **sphere**; this one has radius 2 and center $(0, 0, 0)$.
Comment. Again, this is a 2-dimensional object. (1 equation: $3 - 1 = 2$)

A **sphere** of radius r and center (x_0, y_0, z_0) is described by

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

Why? A sphere is the "skin" (surface, to be more professional) of a ball. In this case, these are all the points (x, y, z) which have distance exactly r from (x_0, y_0, z_0). In equations: $\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r$

Vectors

[Physical objects like force or velocity are vectors...]

Example 19. Given points $P_1 = (1, 1)$ and $P_2 = (3, 2)$, the **vector** $\overrightarrow{P_1P_2}$ is the "arrow" connecting the two points. [The vector measures precisely the displacement from P_1 to P_2.]

- Make a sketch!
- The **component form** of the vector is $\overrightarrow{P_1P_2} = \langle 3 - 1, 2 - 1 \rangle = \langle 2, 1 \rangle$.

  At least for a while, we will use the pointy brackets for vectors, just to distinguish them from points.
- If $O = (0, 0)$ is the origin and P is the point $P = (2, 1)$, then the vector $\overrightarrow{OP} = \langle 2, 1 \rangle$ is the **same vector** as $\overrightarrow{P_1P_2}$ (in our sketch, the corresponding arrows are just in different position).

  [The book says the vector is in "standard position" if we place its tail at the origin.]

Vectors consist of a **length** and a **direction**. (more on direction next time)

- The length of $\boldsymbol{v} = \langle v_1, v_2 \rangle$ is $|\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2}$.
- Likewise, in 3D, the length of $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ is $|\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

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