Example 175.

(a) Determine the Taylor series of $f(x)=\cos(x)$ at $x=0$.

(b) Spell out the Taylor polynomial of order 4 for $f(x)=\cos(x)$ at $x=0$.

Solution.

(a) The derivatives of $f(x)$ cycle through $\cos(x), -\sin(x), -\cos(x), \sin(x), \ldots$.
In particular, the values $f^{(n)}(0)$ cycle through $1, 0, -1, 0, \ldots$.
That is, $f^{(2n)}(0)=(-1)^n$ and $f^{(2n+1)}(0)=0$.
Therefore, the Taylor series of $f(x)=\cos(x)$ at $x=0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Note. Assuming that $\cos x$ can be written as a power series at $x=0$, we conclude that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Again, this can be justified via Taylor's formula or a differential equation.

(b) Truncating the Taylor series, the Taylor polynomial of order 4 is

$$1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Example 176. Determine the Taylor series of $\displaystyle\int e^{-x^2} \mathrm{d}x$ at $x=0$.

Solution. Since $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, it follows that $e^{-x^2} = \displaystyle\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$.

Integrating term by term, we conclude that $\displaystyle\int e^{-x^2} \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} + C$.

Note. Since e^{-x^2} is an even function, its Taylor series only includes the terms x^{2n} (which are even) and not terms of the form x^{2n+1} (which are odd). See also the Taylor series that we got for $\cos(x)$ (which is even).

Armin Straub
straub@southalabama.edu

Example 177. Determine the Taylor series of $f(x)=x^3+3x^2+3x+3$ at $x=-1$.

Solution. By definition, the Taylor series in question is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n = f(-1)+f'(-1)(x+1)+\frac{f''(-1)}{2!}(x+1)^2+\frac{f'''(-1)}{3!}(x+1)^3+\dots$$

Clearly $f(-1)=2$ and we need to compute the other values $f^{(n)}(-1)$ as follows:

- $f'(x)=3x^2+6x+3$ so that $f'(-1)=0$.
- $f''(x)=6x+6$ so that $f''(-1)=0$.
- $f'''(x)=6$ so that $f'''(-1)=6$.
- We note that $f^{(4)}(x)=0$ so that $f^{(n)}(-1)=0$ for all $n\geqslant 4$.

The Taylor series therefore is

$$f(-1)+f'(-1)(x+1)+\frac{f''(-1)}{2!}(x+1)^2+\frac{f'''(-1)}{3!}(x+1)^3=2+\frac{6}{3!}(x+1)^3=2+(x+1)^3.$$

Comment. For a polynomial $f(x)$, the Taylor series is the same polynomial just expanded around a different point. In particular, the Taylor series only has terms up to the degree of $f(x)$.

Example 178. Find a power series (about $x=0$) for $\dfrac{1}{1+x^2}$.

Solution. We plug $-x^2$ for x in the geometric series $\displaystyle\sum_{n=0}^{\infty} x^n=\frac{1}{1-x}$ to get $\displaystyle\sum_{n=0}^{\infty}(-x^2)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n}=\frac{1}{1+x^2}$. This is valid for $|-x^2|<1$ or, equivalently, $|x|<1$.
In particular, this power series has radius of convergence 1.

Example 179. Find a power series (about $x=0$) for $\arctan(x)$.

Solution. Recall that $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x=\arctan(x)+C$.
In Example 178, we observed that $\dfrac{1}{1+x^2}=\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$ and that this power series converges if $|x|<1$.
We now integrate both sides of $\dfrac{1}{1+x^2}=\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$ to find a power series for $\arctan(x)$.

$$\int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,\mathrm{d}x=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}+C$$

Hence, $\arctan(x)=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}+C$. Since $\arctan(0)=0$, it follows that $C=0$.

Example 180. What is the exact interval of convergence in the previous example?

Solution. Since the convergence radius is 1, we know that the series converges for $|x|<1$, and diverges if $|x|>1$. We don't yet know whether the series converges for $x=\pm 1$.

- For $x=1$, we get the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$

  This is an **alternating series** because the terms are alternately positive and negative. Due to the alternating series test, the series converges ($a_n = \frac{1}{2n+1}$ is positive, decreasing and converges to 0).

- For $x=-1$, we get the series $\displaystyle -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ which is -1 times what we get for $x=1$.

  In particular, this series converges as well.

Our conclusion is that the exact interval of convergence is $[-1,1]$.

Comment. The series for $x=\pm 1$ are not **absolutely convergent** because, if we sum instead the absolute values of the terms, then we get $\sum_{n=0}^{\infty} \frac{1}{2n+1} = 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$, and we know that this series diverges, because it is "half" of the harmonic series. This means that the series for $x=\pm 1$ are only **conditionally convergent**.

Since $\arctan(1)=\frac{\pi}{4}$, we conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots = \frac{\pi}{4}$.

Note that $\arctan(-1) = -\arctan(1) = -\frac{\pi}{4}$, which explains why we got the same series times -1.