Example 160. Determine the radius of convergence of the following power series and their exact interval of convergence.

(a) $\displaystyle\sum_{n=0}^{\infty} (n^2+4)\,x^n$ This is a power series about $x=0$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(x-3)^n$ This is a power series about $x=3$.

Solution.

(a) We apply the ratio test with $a_n = (n^2+4)\,x^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((n+1)^2+4)\,x^{n+1}}{(n^2+4)\,x^n}\right| = |x|\,\frac{n^2+2n+5}{n^2+4} \to |x| \text{ as } n\to\infty$$

The ratio test implies that $\displaystyle\sum_{n=0}^{\infty} (n^2+4)\,x^n$ converges if $|x|<1$.

Thus the radius of convergence is 1.

The ratio test does not tell us what happens when $|x|=1$. We now look at those cases more carefully:

- $x=1$: $\displaystyle\sum_{n=0}^{\infty} (n^2+4)$ clearly diverges (because $\displaystyle\lim_{n\to\infty}(n^2+4)$ is not 0).
- $x=-1$: $\displaystyle\sum_{n=0}^{\infty} (n^2+4)(-1)^n$ clearly diverges (because $\displaystyle\lim_{n\to\infty}(n^2+4)(-1)^n$ is not 0).

Combined, $\displaystyle\sum_{n=0}^{\infty} (n^2+4)\,x^n$ converges if and only if x is in $(-1,1)$ (the exact interval of convergence).

(b) **(again, also done last time)** We apply the ratio test with $a_n = \dfrac{2^n}{\sqrt{n}}(x-3)^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+1}}\,\frac{\sqrt{n}}{2^n(x-3)^n}\right| = 2|x-3|\,\sqrt{\frac{n}{n+1}} \to 2|x-3| \text{ as } n\to\infty$$

The ratio test implies that $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(x-3)^n$ converges if $|x-3|<\frac{1}{2}$.

So the radius of convergence is $\frac{1}{2}$.

The ratio test is inconclusive for $|x-3|=\frac{1}{2}$ or, equivalently, $x=3-\frac{1}{2}=\frac{5}{2}$ and $x=3+\frac{1}{2}=\frac{7}{2}$:

- $x=\frac{5}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test ($\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$).

  Comment. We could have also just recognized this as the alternating p-series with $p=\frac{1}{2}$.
- $x=\frac{7}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is the p-series with $p=\frac{1}{2}$ which diverges (because $p\leqslant 1$).

Combined, the exact interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(x-3)^n$ is $\left[\frac{5}{2},\frac{7}{2}\right)$.

Armin Straub
straub@southalabama.edu

Example 161. Let $f(x)=\sum_{n=1}^{\infty}\frac{2^n}{\sqrt{n}}(x-3)^n$ for x with $|x-3|<R$ where $R=\frac{1}{2}$ is the radius of convergence computed in the previous problem. Write down a power series for $f'(x)$.

Solution. $f'(x)=\sum_{n=1}^{\infty}\frac{2^n}{\sqrt{n}}\cdot n\,(x-3)^{n-1}=\sum_{n=1}^{\infty}\sqrt{n}\cdot 2^n\,(x-3)^{n-1}$

Optional. By replacing the index n with $n+1$, we can rewrite this as $f'(x)=\sum_{n=0}^{\infty}\sqrt{n+1}\cdot 2^{n+1}(x-3)^n$.

Example 162. Determine the radius of convergence of $\sum_{n=1}^{\infty}\frac{5^n}{n^2}(4x-3)^{2n}$ (this is a power series about $\frac{3}{4}$) and its exact interval of convergence.

Solution. We apply the ratio test with $a_n=\frac{5^n}{n^2}(4x-3)^{2n}$.

$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{5^{n+1}}{(n+1)^2}(4x-3)^{2n+2}\frac{n^2}{5^n(4x-3)^{2n}}\right|=5|4x-3|^2\frac{n^2}{(n+1)^2}\to 5|4x-3|^2$ as $n\to\infty$

The ratio test implies that $\sum_{n=1}^{\infty}\frac{5^n}{n^2}(4x-3)^{2n}$ converges if $5|4x-3|^2<1$. To focus on x, we can rewrite this as $|4x-3|<\frac{1}{\sqrt{5}}$ or, equivalently, $\left|x-\frac{3}{4}\right|<\frac{1}{4\sqrt{5}}$.

In this latter form, we see that the radius of convergence is $\frac{1}{4\sqrt{5}}$.

The ratio test is inconclusive for $\left|x-\frac{3}{4}\right|=\frac{1}{4\sqrt{5}}$ or, equivalently, $x=\frac{3}{4}-\frac{1}{4\sqrt{5}}$ and $x=\frac{3}{4}+\frac{1}{4\sqrt{5}}$:

- $x=\frac{3}{4}+\frac{1}{4\sqrt{5}}$: $\sum_{n=1}^{\infty}\frac{5^n}{n^2}\left(\frac{1}{\sqrt{5}}\right)^{2n}=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is the p-series with $p=2$ which converges (because $p>1$).
- $x=\frac{3}{4}-\frac{1}{4\sqrt{5}}$: $\sum_{n=1}^{\infty}\frac{5^n}{n^2}\left(-\frac{1}{\sqrt{5}}\right)^{2n}=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is the same series and so converges as well.

Combined, the exact interval of convergence of $\sum_{n=1}^{\infty}\frac{5^n}{n^2}(4x-3)^{2n}$ is $\left[\frac{3}{4}-\frac{1}{4\sqrt{5}},\frac{3}{4}+\frac{1}{4\sqrt{5}}\right]$.

Example 163. What is the radius of convergence of the following power series?

(a) $\sum_{n=0}^{\infty}n!x^n$ (b) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$

Solution.

(a) We apply the ratio test with $a_n=n!x^n$:

$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|=|x|\,(n+1)\to\infty$ as $n\to\infty$ (unless $|x|=0$)

The ratio test implies that $\sum_{n=0}^{\infty}n!x^n$ diverges for all x except $x=0$. The radius of convergence is 0.

(b) We again apply the ratio test, this time with $a_n=\frac{x^n}{n!}$:

$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{n!x^{n+1}}{(n+1)!x^n}\right|=\frac{|x|}{n+1}\to 0$ as $n\to\infty$

The ratio test implies that $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all x. The radius of convergence is ∞.

Comment. In Example 166 below, we find that $\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$.

Example 164. (review/quiz!) Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5\sqrt{n}+n^2}{3n^3+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{5\sqrt{n}+2\ln(n)}{3n^3+1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{5\sqrt{n}+2n^3}{3n^3+1}$

Solution.

(a) Diverges by limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3n^3} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series which diverges).

(b) Converges by limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{5\sqrt{n}}{3n^3} = \frac{5}{3}\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ (p-series with $p=5/2>1$ which converges).

(c) Diverges by n-th term test since $\dfrac{5\sqrt{n}+2n^3}{3n^3+1} \to \dfrac{2}{3}$ as $n \to \infty$.