**Review.** Ratio test

**Example 152.** Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{n!}$

**Solution.**

(a) We apply the ratio test with $a_n = \frac{n^2}{2^n}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{2^{n+1}} \frac{2^n}{n^2} = \frac{1}{2} \frac{(n+1)^2}{n^2} = \frac{1}{2} \frac{n^2+2n+1}{n^2} \to \frac{1}{2}$ as $n \to \infty$

Since $\frac{1}{2} < 1$, the ratio test implies that $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

(b) Note that $\frac{2^n}{n^3} \to \infty \neq 0$. Hence, the series diverges.

**Alternatively.** Suppose we didn't realize this and, instead, we apply the ratio test with $a_n = \frac{2^n}{n^3}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^3} \frac{n^3}{2^n} = 2 \frac{n^3}{(n+1)^3} = 2 \frac{n^3}{n^3+3n^2+3n+1} \to 2$ as $n \to \infty$

Since $2 > 1$, the ratio test implies that $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges.

(c) We apply the ratio test with $a_n = \frac{(-1)^n 5^n}{n!}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{5^{n+1}}{(n+1)!} \frac{n!}{5^n} = \frac{5}{n+1} \to 0$ as $n \to \infty$

Since $0 < 1$, the ratio test implies that $\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{n!}$ converges.

**Review.** Recall that $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. This is the **factorial**.

It counts the number of ways in which you can order $n$ objects.

## The alternating series test

**Theorem 153. (Alternating series test)** If $a_n$ is positive, decreasing with $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=N}^{\infty} (-1)^n a_n$ converges.

**Why?** Proof by picture!

**Example 154.** Does the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge?

**Solution.** Yes, it converges by the alternating series test: $a_n = \frac{1}{n}$ is positive, decreasing, and $\lim_{n\to\infty} a_n = 0$.

**Important.** Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not **converge absolutely**.

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**Example 155.** For which $p$ does the alternating $p$-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge? For which $p$ does it converge absolutely?

**Solution.**

- If $p > 0$, then the series converges by the alternating series test, because $a_n = \frac{1}{n^p}$ is positive, decreasing, and $\lim_{n\to\infty} a_n = 0$. If $p \leqslant 0$, then $\lim_{n\to\infty} \frac{(-1)^n}{n^p}$ is not zero. Therefore, the series diverges.
- By definition, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges absolutely if and only if $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^p}\right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. Since this is just the usual $p$-series, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges absolutely if and only if $p > 1$.

In summary, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges if and only if $p > 0$, and converges absolutely if and only if $p > 1$.

## Power series

**Definition 156.** A **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Note that this is like a polynomial of infinite degree. This polynomial is "centered at" $x = a$ so that, if $x$ is close to $a$ (meaning that $x - a$ is close to zero), then the terms $(x-a)^n$ are getting smaller and smaller as $n \to \infty$.
In particular, a **power series** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

We will soon see how to write familiar functions like $e^x, \cos(x), \dots$ as power series. The next example gives a preview.

**Example 157.** Determine all $x$ for which the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges.

**Solution.** We apply the ratio test with $a_n = \frac{x^n}{n}$.
$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1}\frac{n}{x^n}\right| = |x|\frac{n}{n+1} \to |x|$ as $n \to \infty$
The ratio test implies that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges if $|x| < 1$ (and diverges if $|x| > 1$).
Note that the ratio test does not tell us what happens when $|x| = 1$. We need to look at those cases more carefully:

- $x = 1$: In this case we get the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we know diverges.
- $x = -1$: In this case we get the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which we know converges.

In summary, the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges if and only if $x$ is in $[-1, 1)$.

**Note.** This is a power series around $0$. The series converges for all $x$ less than $1$ away from $0$. This is why we say that the power series has **radius of convergence** $1$.

**Looking ahead.** Which function is hiding behind $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$?

Power series can be differentiated term-by-term and so $f'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n}=\sum_{n=1}^{\infty}x^{n-1}=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$.

Therefore, $f(x)=\int\frac{1}{1-x}\mathrm{d}x=-\ln|1-x|+C$.

Finally, the fact that $f(0)=0$ implies that $C=0$. This means that $f(x)=-\ln|1-x|$.

Note that this function is nice at $x=0$ (and around it) but that it has a problem for $x=1$. Since converging power series do not have problems, this is one way to see that the radius of convergence is $1$.

Let us again look at the special values $x=1$ and $x=-1$. First, we clearly have $\lim_{x\to 1}f(x)=+\infty$.

On the other hand, $\lim_{x\to -1}f(x)=-\ln(2)$ implying that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\ldots=-\ln(2)\approx -0.693$

**Theorem 158.** Every power series $\sum_{n=0}^{\infty}c_n(x-a)^n$ has a **radius of convergence** $R$, meaning:

(a) if $R=0$, then the series converges only for $x=a$,

(b) if $0<R<\infty$, then the series converges for all $x$ such that $|x-a|<R$

but diverges if $|x-a|>R$ (in other words, $R$ is as large as possible),

(c) if $R=\infty$, then the series converges for all $x$.

Note that, if $0<R<\infty$, no general statement can be made for the case $|x-a|=R$.

The exact **interval of convergence** can be $(a-R,a+R)$ or $[a-R,a+R)$ or $(a-R,a+R]$ or $[a-R,a+R]$.

## Example 159.

(a) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty}\frac{2^n}{\sqrt{n}}(x-3)^n$.

(b) Determine exactly for which values of $x$ this series converges. In other words, what is the exact interval of convergence of this series?

**Solution.** Note that this is a power series about $x=3$.

(a) We apply the ratio test with $a_n=\frac{2^n}{\sqrt{n}}(x-3)^n$.

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{2^n(x-3)^n}\right|=2|x-3|\sqrt{\frac{n}{n+1}}\to 2|x-3| \text{ as } n\to\infty$$

The ratio test implies that $\sum_{n=1}^{\infty}\frac{2^n}{\sqrt{n}}(x-3)^n$ converges if $|x-3|<\frac{1}{2}$. So the radius of convergence is $\frac{1}{2}$.

(b) The ratio test is inconclusive for $|x-3|=\frac{1}{2}$ or, equivalently, $x=3-\frac{1}{2}=\frac{5}{2}$ and $x=3+\frac{1}{2}=\frac{7}{2}$:

- $x=\frac{5}{2}$: $\sum_{n=1}^{\infty}\frac{2^n}{\sqrt{n}}\left(-\frac{1}{2}\right)^n=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test ( $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$).
  **Comment.** We could have also just recognized this as the alternating $p$-series with $p=\frac{1}{2}$.
- $x=\frac{7}{2}$: $\sum_{n=1}^{\infty}\frac{2^n}{\sqrt{n}}\left(\frac{1}{2}\right)^n=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is the $p$-series with $p=\frac{1}{2}$ which diverges (because $p\leqslant 1$).

Combined, $\sum_{n=1}^{\infty}\frac{2^n}{\sqrt{n}}(x-3)^n$ converges if and only if $x$ is in $\left[\frac{5}{2},\frac{7}{2}\right)$ (the exact interval of convergence).