Review. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Direct comparison and limit comparison

Recall that we discussed direct comparison and limit comparison tests for improper integrals. The same ideas apply to series:

We assume that both $a_n \geqslant 0$ and $b_n \geqslant 0$.

(direct comparison)

- $\sum_{n=N}^{\infty} a_n$ converges if $a_n \leqslant b_n$ and $\sum_{n=N}^{\infty} b_n$ converges.
- $\sum_{n=N}^{\infty} a_n$ diverges if $a_n \geqslant b_n$ and $\sum_{n=N}^{\infty} b_n$ diverges.

(limit comparison)

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ for $0 < L < \infty$ then $\sum_{n=N}^{\infty} a_n$ and $\sum_{n=N}^{\infty} b_n$ both converge or both diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=N}^{\infty} b_n$ converges, then $\sum_{n=N}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=N}^{\infty} b_n$ diverges, then $\sum_{n=N}^{\infty} a_n$ diverges.

Example 148. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2 + \ln(n)}$

(b) $\sum_{n=1}^{\infty} \frac{n + \ln(n)}{3n^2 + 4}$

(c) $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+3}$

(d) $\sum_{n=1}^{\infty} \frac{2\sqrt{n}+1}{3n^2+3}$

(e) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$

Solution.

(a) **(direct comparison)** Note that, for all $n \geqslant 1$, $n^2 + \log(n) \geqslant n^2$ and so $\frac{1}{n^2+\log(n)} \leqslant \frac{1}{n^2}$.

By comparison, $\sum_{n=1}^{\infty} \frac{1}{n^2+\log(n)} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} = \text{finite}$ and so $\sum_{n=1}^{\infty} \frac{1}{n^2+\log(n)}$ converges.

(limit comparison) Observe that we can just "see" this: for large n, our terms $\frac{1}{n^2+\log(n)}$ "behave" like $\frac{1}{n^2}$ and so $\sum_{n=1}^{\infty} \frac{1}{n^2+\log(n)}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. This reasoning is made precise by the limit comparison test. Spell out the details!

Comment. Like in this example, we often have a "natural" comparison with a geometric series or a p-series. In those cases, don't spend time thinking about the corresponding integral! Here, $\int_1^{\infty} \frac{\mathrm{d}x}{x^2+\log(x)}$ is such that its antiderivative cannot even be written in terms of the functions we are familiar with.

(b) Our terms $a_n = \frac{n+\ln(n)}{3n^2+4}$ behave like $\frac{n}{3n^2} = \frac{1}{3n}$ for large n. Thus, we should do limit comparison with $b_n = \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(n+\ln(n))}{3n^2+4} = \frac{1}{3}.$$

This means that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (this is the harmonic series), our series diverges as well.

(c) We do a limit comparison of the sequence $a_n = \frac{2n+1}{3n^2+3}$ with $b_n = \frac{1}{n}$.

First, we check that $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2}{3}$. By the limit comparison, we then find that $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+3}$ diverges because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(d) We do a limit comparison with $b_n = \frac{1}{n^{3/2}}$. Since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges ($3/2 > 1$), we conclude that $\sum_{n=1}^{\infty} \frac{2\sqrt{n}+1}{3n^2+3}$ converges as well.

(e) Since $0 \leqslant \sin^2(n) \leqslant 1$, the series $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$ converges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (which converges since it is a p-series with $p = 2 > 1$).

Frequently, we have choices which test to apply to determine whether a series converges:

Example 149. Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution. (direct comparison) Note that $\frac{\ln n}{n} > \frac{1}{n}$ for all $n > 3$ (because $\ln n > 1$ for $n > 3$).

But already $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (note that $\sum_{n=3}^{\infty} \frac{\ln n}{n} > \sum_{n=3}^{\infty} \frac{1}{n}$), so $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ has to diverge as well.

Solution. (limit comparison) We do limit comparison of $a_n = \frac{\ln n}{n}$ and $b_n = \frac{1}{n}$ so that $\frac{a_n}{b_n} = \ln n$. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, we conclude that $\sum_{n=N}^{\infty} a_n$ diverges as well.

Solution. (via integral comparison) Note that $\frac{\ln x}{x}$ is decreasing for $x \geqslant 3$. (Can you show why?) The integral $\int_3^{\infty} \frac{\ln x}{x}\,\mathrm{d}x = \int_{\ln(3)}^{\infty} u\,\mathrm{d}u$ obviously diverges (note that we substituted $u = \ln x$), and hence $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges as well.

Comment. Usually, it is a good idea to avoid integral comparison unless there are no easier options. Even here, we had to be a bit careful since $\ln x / x$ does not decrease for all $x \geqslant 1$.

The ratio test

(absolute convergence)

We say that the series $\sum_{n=N}^{\infty} a_n$ **converges absolutely** if $\sum_{n=N}^{\infty} |a_n|$ converges.

It is not hard to see (check out Section 9.5 in the book) that absolute convergence implies (regular) convergence.

It is often easier to work with series where all terms are $\geqslant 0$. Therefore, it is often easier to establish absolute convergence of a series (and then get convergence for free). Some tests like the **ratio test** even give us absolute convergence.

Caution! There are series which converge but which do not converge absolutely.

One example is the alternating harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$. We will discuss **alternating series** soon.

Theorem 150. (Ratio test) Suppose the limit $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists.

- If $L<1$, then the series $\sum_{n=N}^{\infty} a_n$ converges (absolutely).
- If $L>1$, then the series $\sum_{n=N}^{\infty} a_n$ diverges.
- If $L=1$, then we don't know. The test is inconclusive.

Note that a series $\sum_{n=N}^{\infty} a_n$ is geometric if $\frac{a_{n+1}}{a_n}=L$ is constant. It converges if and only if $|L|<1$.

Example 151. Apply the ratio test to the geometric series $\sum_{n=0}^{\infty} x^n$.

Solution. In this case, $a_n=x^n$ and so $\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x^{n+1}}{x^n}\right|=|x|$.

The ratio test with $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|x|$ then shows that $\sum_{n=0}^{\infty} x^n$ converges if $|x|<1$, and diverges if $|x|>1$.

Important. The ratio test makes no statement about the cases $x=1$ and $x=-1$. In these cases, we need to do additional analysis. Here, it is easy to see directly that the geometric series diverges when $x=1$ or $x=-1$.