Example 142. (review) Determine the value of the following series or state that they diverge.

(a) $\sum_{n=1}^{\infty} \frac{2^n - 5\cdot 3^n}{5^n}$ (b) $\sum_{n=1}^{\infty} \frac{2^n - 3\cdot 5^n}{5^n}$ (c) $\sum_{n=2}^{\infty} \frac{3n^2+2}{5n^2+1}$

Solution.

(a) $\sum_{n=1}^{\infty} \frac{2^n - 5\cdot 3^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n - 5\cdot \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n = \left(\frac{1}{1-\frac{2}{5}} - 1\right) - 5\left(\frac{1}{1-\frac{3}{5}} - 1\right) = \frac{2}{3} - 5\cdot\frac{3}{2} = -\frac{41}{6}$

(b) $\sum_{n=1}^{\infty} \frac{2^n - 3\cdot 5^n}{5^n}$ diverges (by the n-th term test) because $\frac{2^n - 3\cdot 5^n}{5^n} \to -3 \neq 0$ as $n \to \infty$.

(c) $\sum_{n=2}^{\infty} \frac{3n^2+2}{5n^2+1}$ diverges (by the n-th term test) because $\frac{3n^2+2}{5n^2+1} \to \frac{3}{5} \neq 0$ as $n \to \infty$.

Integral comparison test

Theorem 143. (Integral comparison test) Suppose that the function $f(x)$ is positive, continuous and decreasing for $x \geqslant N$. Then:

$$\sum_{n=N}^{\infty} f(n) \text{ converges} \iff \int_N^{\infty} f(x)\mathrm{d}x \text{ converges}$$

In other words, the series and integral both converge or both diverge.

Why? Make a sketch where you compare the area under the curve $f(x)$ with rectangles of width 1. (See Section 9.3 in our book for nice illustrations.)

Warning: if they converge, of course, the values of the series and the integral are going to be different!

Example 144. The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges. Why?

Solution. Note that $\lim_{n\to\infty} \frac{1}{n} = 0$, so we cannot directly use our test for divergence coming out of Theorem 140.

However, we can combine terms as follows to see the divergence:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geqslant \frac{1}{4}+\frac{1}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geqslant \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geqslant \frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16} = \frac{1}{2}} + \ldots$$

Solution. (integral comparison test) The function $f(x) = \frac{1}{x}$ is positive, continuous and decreasing for $x \geqslant 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ converges} \iff \int_1^{\infty} \frac{1}{x}\mathrm{d}x \text{ converges.}$$

Since $\int_1^M \frac{1}{x}\mathrm{d}x = \Big[\ln|x|\Big]_1^M = \ln M \to \infty$ as $M \to \infty$, the integral diverges. It follows, by comparison, that the harmonic series diverges, too.

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Example 145. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ converges.

[It is considerably more difficult to show that, in fact, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.]

Solution. As in the previous example, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges if and only if the integral $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$ converges. Since $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x = \left[-\frac{1}{x}\right]_1^{\infty} = 0 - (-1) = 1$, the integral converges, and so the series converges as well.

More generally, we have the following result:

(p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a **p-series**. It converges if and only if $p > 1$.

Why? This follows from the integral comparison test, and because $\int_1^{\infty} \frac{\mathrm{d}x}{x^p}$ converges if and only if $p > 1$. See next example.

Example 146. For what values of p does $\int_1^{\infty} \frac{\mathrm{d}x}{x^p}$ converge?

Solution. For $p \neq 1$, we have $\int_1^{\infty} \frac{\mathrm{d}x}{x^p} = \left[\frac{1}{-p+1} x^{-p+1}\right]_1^{\infty}$.

If $p > 1$, then $\lim_{x\to\infty} x^{-p+1} = \lim_{x\to\infty} \frac{1}{x^{p-1}} = 0$, and we find that the integral converges.

If $p < 1$, then $\lim_{x\to\infty} x^{-p+1} = \infty$, and we find that the integral diverges.

We are missing only the case $p = 1$: in that case, $\int_1^{\infty} \frac{1}{x}\,\mathrm{d}x = \left[\ln|x|\right]_1^{\infty}$ diverges because $\lim_{x\to\infty} \ln|x| = \infty$.

In summary, $\int_1^{\infty} \frac{\mathrm{d}x}{x^p}$ converges if and only if $p > 1$.

Example 147. Determine whether the following series converge or diverge.

(a) $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ (b) $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

Solution. You can use the integral comparison test to find that the first series diverges and the second series converges. However, it is more convenient to look at these series by comparison which is what we discuss next.