Review. If $|x|<1$, then (and only then) the geometric series converges and we have

$$\sum_{n=0}^{\infty} x^n = 1+x+x^2+\ldots = \frac{1}{1-x}.$$

If we have a geometric series that does not start at $n=0$, we can either adjust for the missing terms (by subtracting them off after using the formula for $n \geqslant 0$) or we can pull out a power of x. For instance:

$$\sum_{n=2}^{\infty} x^n = \frac{1}{1-x} - 1 - x, \quad \sum_{n=2}^{\infty} x^n = x^2+x^3+x^4+\ldots = x^2 \sum_{n=0}^{\infty} x^n = \frac{x^2}{1-x}.$$

The end result is the same (make sure you see that) and it is up to us to choose whichever we feel is simpler for a given problem. Below, we usually choose the former approach.

Example 135. (review) Compute the following series (or state that it diverges):

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{5}{3^n}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{5^n}{3^n}$ (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$

Solution.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{5}{3^n} = 5\sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = 5\left(\frac{1}{1-\frac{1}{3}} - \left(\frac{1}{3}\right)^0 - \left(\frac{1}{3}\right)^1\right) = 5\left(\frac{3}{2} - 1 - \frac{1}{3}\right) = \frac{5}{6}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{5^n}{3^n} = \sum_{n=2}^{\infty} \left(\frac{5}{3}\right)^n$ diverges because $\left|\frac{5}{3}\right| \geqslant 1$.

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$ provided that $|-x^2|<1$ (which is the same as $|x|<1$). If this condition is not true, then the series diverges.

Example 136. Write the series $e^{2x}+e^{3x}+e^{4x}+\ldots$ using Σ-notation and evaluate it.

Solution. $e^{2x}+e^{3x}+e^{4x}+\ldots = \displaystyle\sum_{n=2}^{\infty} e^{nx} = \sum_{n=0}^{\infty} e^{nx} - 1 - e^x = \frac{1}{1-e^x} - 1 - e^x$

Here, we need to assume that $|e^x|<1$ (or, equivalently, $x<0$).

Alternatively. $e^{2x}+e^{3x}+e^{4x}+\ldots = e^{2x}(1+e^x+e^{2x}+\ldots) = e^{2x}\displaystyle\sum_{n=0}^{\infty} e^{nx} = \frac{e^{2x}}{1-e^x}$

Check that this is the same!

Example 137. Express the number $0.7777\ldots$ as a rational number.

Solution. (using geometric series)

$$0.7777\ldots = \frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+\ldots = \sum_{n=1}^{\infty} \frac{7}{10^n} = 7\left(\frac{1}{1-\frac{1}{10}} - 10^0\right) = 7\left(\frac{10}{9}-1\right) = \frac{7}{9}$$

Solution. (highschool) Everyone is familiar with $0.3333\ldots = \frac{1}{3}$. This implies that $0.1111\ldots = \frac{1}{3}\cdot 0.3333\ldots = \frac{1}{9}$. Hence, our number is $0.7777\ldots = 7\cdot 0.1111\ldots = \frac{7}{9}$.

Example 138. Express the number $2.313131\ldots$ as a rational number.

Solution.

$$2.313131\ldots = 2+\frac{31}{100}+\frac{31}{(100)^2}+\frac{31}{(100)^3}+\ldots = 2+31\sum_{n=1}^{\infty}\frac{1}{100^n} = 2+31\left(\frac{1}{1-\frac{1}{100}}-\frac{1}{100^0}\right) = 2+\frac{31}{99} = \frac{229}{99}$$

The previous two examples teach us something fundamental about numbers:

Rational numbers are precisely those numbers which have a finite (like 1.5) or repeating (like $2.313131\ldots$) decimal expansion.

Moreover, there is some ambiguity because finite decimals, like 1.5, can also be written in the repeating fashion $1.5 = 1.4999\ldots$.

As a consequence, irrational numbers like $\sqrt{2}$ or π never have a repeating decimal expansion.

Example 139. Explain the following:

$$\frac{1}{499} = 0.002004008016032064128256\ldots$$

Solution. The thing to explain is that there appear to be powers of 2 in the decimal on the right-hand side. Note that the right-hand side can be realized as a geometric series:

$$\frac{2}{1000}+\left(\frac{2}{1000}\right)^2+\left(\frac{2}{1000}\right)^3+\ldots = \frac{1}{1-\frac{2}{1000}}-1 = \frac{1}{499}.$$

The nth-term test for divergence

Review. Recall that the improper integral $\int_N^{\infty} f(x)\mathrm{d}x$ converges if and only if the limit $\lim_{M\to\infty}\int_N^M f(x)\mathrm{d}x$ exists. Likewise, $\sum_{n=N}^{\infty} a_n$ converges if and only if the limit $\lim_{M\to\infty}\sum_{n=N}^{M} a_n$ exists.

For the series $\sum_{n=N}^{\infty} a_n$ to converge, it is necessary that $\lim_{n\to\infty} a_n = 0$.

Can you explain how this follows from the definition of limit?

Intuitively, this is simply saying that the only hope to be able to add infinitely many things (and get something finite) is if these things are very small.

Theorem 140. (nth-term test for divergence) If $\lim_{n\to\infty} a_n$ is not 0, then $\sum_{n=1}^{\infty} a_n$ diverges.

(In particular, if the limit $\lim_{n\to\infty} a_n$ does not exist, then the series diverges.)

Example 141. Show that the following series all diverge.

(a) $\sum_{n=0}^{\infty}\frac{3^n+5^n}{10\cdot 5^n}$ (b) $\sum_{n=1}^{\infty}(-1)^n$ (c) $\sum_{n=1}^{\infty}\frac{n^2}{3n^2+7}$ (d) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{\log(n)}$

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Solution.

(a) Note that $\lim_{n\to\infty} \frac{3^n+5^n}{10\cdot 5^n} = \frac{1}{10} \neq 0$. Hence, the series diverges by the nth-term test for divergence.

(b) The sequence $(-1)^n$ does not converge to 0 as $n\to\infty$. Hence, the series diverges by the nth-term test.

(c) Since $\lim_{n\to\infty} \frac{n^2}{3n^2+7} = \frac{1}{3} \neq 0$ the series diverges by the nth-term test for divergence.

(d) This series diverges by the nth-term test for divergence because $\lim_{n\to\infty} \frac{\sqrt{n}}{\log n}$ is not zero.
In fact, $\lim_{n\to\infty} \frac{\sqrt{n}}{\log n} = \infty$. (If you don't see this, apply L'Hospital!)

A word of caution. The nth-term test for divergence only gives a necessary condition. It is not sufficient!

For instance, as we will discuss next, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges although $\lim_{n\to\infty} \frac{1}{n} = 0$.