Example 130. (Halloween scare!) Let $a = b$. Then $a^2 = ab$, so $a^2 + a^2 = a^2 + ab$ or $2a^2 = a^2 + ab$. Hence, $2a^2 - 2ab = a^2 - ab$ or $2(a^2 - ab) = a^2 - ab$. Cancelling, we arrive at $2 = 1$.

[Can you see the foul but disguised division by zero?!]

Example 131. (Halloween scare!) Foul play with divergent series:

$$\begin{aligned} 0 &= (1-1)+(1-1)+(1-1)+\dots \\ &= 1-1+1-1+1-1+\dots \\ &= 1+(-1+1)+(-1+1)+\dots \\ &= 1+0+0\dots \\ &= 1 \end{aligned}$$

Where did this go wrong? Note that in the second line we have the series $\sum_{n=0}^{\infty} (-1)^n$ which is divergent.

Lesson. Divergent series don't conform to our usual laws.

In other words, convergence of series is crucially important when working with them.

The fallacy above is somewhat similar to the argument "$\infty = \infty + 1$, so $0 = 1$".

Example 132. (geometric series, preview) To follow-up the previous example on a positive note: if a series converges, then we can work with it. The following argument for evaluating the geometric series is valid provided that the series converges (which we soon see is if $|x| < 1$):

$$S = 1 + x + x^2 + x^3 + \dots \quad \rightsquigarrow \quad xS = x + x^2 + x^3 + x^4 + \dots = S - 1 \quad \rightsquigarrow \quad S = \frac{1}{1-x}$$

Series

A tortoise racing a Greek hero... **Zeno's paradox**:

`https://en.wikipedia.org/wiki/Zeno%27s_paradoxes#Achilles_and_the_tortoise`

Example 133. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

Solution. Visual!

Solution. Redo this example by taking the limit of a geometric sum.

Geometric series

Review. The geometric sum is

$$\sum_{n=0}^{M} x^n = 1 + x + x^2 + \dots + x^M = \frac{1 - x^{M+1}}{1 - x}.$$

Armin Straub
straub@southalabama.edu

Taking the limit $M \to \infty$ in the geometric sum, we get: (recall that $\lim_{M\to\infty} x^M = 0$ if $|x|<1$)

(geometric series) If $|x|<1$, then

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x}.$$

If $|x| \geqslant 1$, then the geometric series diverges.

Example 134. Compute the following series (or state that it diverges):

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{2^n}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{7}{10^n}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{7}{10^n}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{5}{3^n}$

(f) $\displaystyle\sum_{n=2}^{\infty} 3\cdot 4^{-n}$

(g) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{7}{2^n} - \frac{3^n}{5^n}\right)$

(h) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{3^n}$

(i) $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$

Solution.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} - \left(\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2}\right) = 2 - \left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{1}{4}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{7}{10^n} = 7\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = 7\cdot\frac{1}{1-\frac{1}{10}} = \frac{70}{9}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{7}{10^n} = \sum_{n=0}^{\infty} \frac{7}{10^n} - \left(\frac{7}{10^0} + \frac{7}{10^1}\right) = \frac{70}{9} - \left(7 + \frac{7}{10}\right) = \frac{7}{90}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{5}{3^n} = 5\sum_{n=0}^{\infty} \frac{1}{3^n} = 5\cdot\frac{1}{1-\frac{1}{3}} = \frac{15}{2}$

(f) $\displaystyle\sum_{n=2}^{\infty} 3\cdot 4^{-n} = 3\sum_{n=2}^{\infty} \frac{1}{4^n} = 3\left(\frac{1}{1-\frac{1}{4}} - 1 - \frac{1}{4}\right) = \frac{1}{4}$

(g) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{7}{2^n} - \frac{3^n}{5^n}\right) = 7\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = 7\cdot\frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{3}{5}} = 14 - \frac{5}{2} = \frac{23}{2}$

(h) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{5}{3}\right)^n$ doesn't converge because $\left|\frac{5}{3}\right| \geqslant 1$.

(i) $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$ provided that $|-x^2| < 1$ (which is the same as $|x|<1$). If this condition is not true, then the series diverges.

The very last example illustrates an important point. Namely, it shows that there is a novel way to think about (and get our hands on) functions like $\frac{1}{1+x^2}$.

Recall that we care about this function in particular, because it was a building block in partial fractions. For instance, we know that its antiderivative is $\arctan(x)$.

This is the main reason why we are learning about series in a course that focuses on functions!

We will see that it is very convenient to work with series representing functions: they can be differentiated and integrated, and give us an opportunity to work with functions that cannot be written in terms of the "usual" functions.