Limits of sequences

If $\lim_{x\to\infty} f(x) = L$ (the limit of a function: x is real)
then $\lim_{n\to\infty} f(n) = L$ (the limit of a sequence: n is an integer).

Important. The reverse is not true: for instance, $\lim_{n\to\infty} \sin(\pi n) = 0$ but $\lim_{x\to\infty} \sin(\pi x)$ does not exist.

Example 125. Determine the following limits:

(a) $\lim_{n\to\infty} \frac{3n^2+7n-8}{8n^2+n+1}$

(b) $\lim_{n\to\infty} \frac{3n^2+7n-8}{8n^3+n+1}$

(c) $\lim_{n\to\infty} \frac{\sqrt{7n^2+4n^6}+5n^3}{3+8n^2+2n^3}$

(d) $\lim_{n\to\infty} \frac{e^{3n}}{n^2}$

(e) $\lim_{n\to\infty} \frac{\sin(n)}{n}$

(f) $\lim_{n\to\infty} \cos\left(\pi - \frac{1}{n^2}\right)$

Solution. In each example, try to first "see" the limit! Then, apply some technique (like L'Hospital) to confirm.

(a) $\lim_{n\to\infty} \frac{3n^2+7n-8}{8n^2+n+1} = \frac{3}{8}$

The easiest way to see this is to note that the main terms in the numerator and denominator are $3n^2$ and $8n^2$. It follows that the limit is the same as $\lim_{n\to\infty} \frac{3n^2}{8n^2} = \lim_{n\to\infty} \frac{3}{8} = 0$.

We can make this argument precise in two different ways:

- $\frac{3n^2+7n-8}{8n^2+n+1} = \frac{3+\frac{7}{n}-\frac{8}{n^2}}{8+\frac{1}{n}+\frac{1}{n^2}}$ and now we can observe that all terms like $7/n$ go to 0 as $n\to\infty$.
- Since the quotient is of the undetermined form "$\frac{\infty}{\infty}$", we can apply L'Hospital (twice):

$$\lim_{n\to\infty} \frac{3n^2+7n-8}{8n^2+n+1} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{6n+7}{16n+1} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{6}{16} = \frac{3}{8}$$

(b) $\lim_{n\to\infty} \frac{3n^2+7n-8}{8n^3+n+1} = \lim_{n\to\infty} \frac{3n^2}{8n^3} = \lim_{n\to\infty} \frac{3}{8n} = 0$

(c) $\lim_{n\to\infty} \frac{\sqrt{7n^2+4n^6}+5n^3}{3+8n^2+2n^3} = \lim_{n\to\infty} \frac{\sqrt{4n^6}+5n^3}{2n^3} = \lim_{n\to\infty} \frac{2n^3+5n^3}{2n^3} = \frac{7}{2}$

(d) $\lim_{n\to\infty} \frac{e^{3n}}{n^2} = \infty$

This is clear if you keep in mind that exponential growth exceeds any polynomial growth.

If needed, we can apply L'Hospital (twice!) since the limit is of the form "$\frac{\infty}{\infty}$":

$$\lim_{n\to\infty} \frac{e^{3n}}{n^2} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{3e^{3n}}{2n} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{9e^{3n}}{2} = \infty$$

(e) $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

Note that $-\frac{1}{n} \leqslant \frac{\sin(n)}{n} \leqslant \frac{1}{n}$. Since our sequence is squeezed between two sequences which approach 0, our limit has to be 0 as well.

Important. We cannot apply L'Hospital because the limit is not of the form "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$". If we did, we would get the limit $\lim_{n\to\infty} \frac{\cos(n)}{1}$ which does not exist (because the values oscillate between -1 and 1).

(f) $\lim_{n\to\infty} \cos\left(\pi - \frac{1}{n^2}\right) = \cos(\pi) = -1$