

Example 110. (review) Evaluate $\int \frac{3x-4}{(x-2)(x+1)(x+2)} dx$ by partial fractions.

Solution. We don't need to do long division first because the numerator is of degree lower than the denominator.

Partial fractions tells us that $\frac{3x-4}{(x-2)(x+1)(x+2)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x+2}$ for certain numbers A, B, C .

- We multiply both sides with $(x-2)(x+1)(x+2)$ to clear denominators:

$$3x-4 = (x+1)(x+2)A + (x-2)(x+2)B + (x-2)(x+1)C$$

- Set $x=2$ to get $2=12A$ so that $A=\frac{1}{6}$.
Set $x=-1$ to get $-7=-3B$ so that $B=\frac{7}{3}$.
Set $x=-2$ to get $-10=4C$ so that $C=-\frac{5}{2}$.
- Consequently: $\frac{3x-4}{(x-2)(x+1)(x+2)} = \frac{1}{6} \cdot \frac{1}{x-2} + \frac{7}{3} \cdot \frac{1}{x+1} - \frac{5}{2} \cdot \frac{1}{x+2}$.

We can therefore compute the integral as:

$$\int \frac{3x-4}{(x-2)(x+1)(x+2)} dx = \frac{1}{6} \int \frac{dx}{x-2} + \frac{7}{3} \int \frac{dx}{x+1} - \frac{5}{2} \int \frac{dx}{x+2} = \frac{1}{6} \ln|x-2| + \frac{7}{3} \ln|x+1| - \frac{5}{2} \ln|x+2| + C$$

Example 111. Determine the shape (but not the exact numbers involved) of the partial fraction decomposition of the following rational functions.

(a) $\frac{x^2-2}{x^4-x^2}$

(c) $\frac{x^3-7x+1}{x^2(x^2+1)}$

(b) $\frac{x^7-2}{x^4-x^2}$

(d) $\frac{x^2+5}{(x+2)^3(x^2+1)^2}$

Solution.

(a) $\frac{x^2-2}{x^4-x^2} = \frac{x^2-2}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}$

- (b) Note that in this case, we need to do long division first. Since $x^7/x^4 = x^3$, the result is of the form $Ax^3 + Bx^2 + Cx + D$ with some remainder that still needs to be divided by $x^4 - x^2$. Hence:

$$\frac{x^7-2}{x^4-x^2} = \frac{x^7-2}{x^2(x-1)(x+1)} = Ax^3 + Bx^2 + Cx + D + \frac{E}{x} + \frac{F}{x^2} + \frac{G}{x-1} + \frac{H}{x+1}$$

(c) $\frac{x^3-7x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

(d) $\frac{x^2+5}{(x+2)^3(x^2+1)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$

Example 112. Evaluate $\int \frac{2x+1}{x^2+6x+9} dx$.

Solution. We don't need to do long division first because the numerator is of degree lower than the denominator.

- This time, after factoring, partial fractions tells us that

$$\frac{2x+1}{x^2+6x+9} = \frac{2x+1}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}.$$

- Clearing denominators, $2x+1 = (x+3)A+B$. Setting $x = -3$, we find $-5 = B$. There is no "magic" next choice for x so we just set $x = 0$ (any other choice works as well) to get $1 = 3A - 5$, which implies $A = 2$.
- Integration now is again straightforward (make sure it is to you!):

$$\int \frac{2x+1}{x^2+6x+9} dx = \int \frac{2}{x+3} dx + \int \frac{-5}{(x+3)^2} dx = 2\ln|x+3| + \frac{5}{x+3} + C.$$

Alternative. While there is no "magic" next choice for x , we can take derivatives to get rid of B . Indeed, differentiating $2x+1 = (x+3)A+B$ gives $2 = A$ directly.

Comment. The form $\frac{A}{x+3} + \frac{B}{(x+3)^2}$ is equivalent to the form $\frac{Cx+D}{(x+3)^2}$. However, the former is more useful for integrating.

Example 113. (homework) Evaluate $\int \frac{x^7+3x+1}{x^4+x^2} dx$.

Solution.

- Since the degree of the numerator is not less than the degree of the denominator, we first perform long division. In this case, we get

$$\frac{x^7+3x+1}{x^4+x^2} = x^3 - x + \frac{x^3+3x+1}{x^4+x^2}.$$

- For the remainder part, partial fractions now tells us its decomposed shape:

$$\frac{x^3+3x+1}{x^4+x^2} = \frac{x^3+3x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

- We multiply both sides with $x^2(x^2+1)$ to clear denominators:

$$x^3+3x+1 = x(x^2+1)A + (x^2+1)B + x^2(Cx+D)$$

- We can now compare the coefficients of $x^3, x^2, x, 1$ on both sides.

Coefficients of x^3 : $1 = A + C$

Coefficients of x^2 : $0 = B + D$

Coefficients of x : $3 = A$

Coefficients of 1 : $1 = B$.

Hence, $A = 3$, $B = 1$, $C = 1 - A = -2$, $D = -B = -1$.

Note. By coefficient of 1 we mean the constant terms of the polynomials (the stuff without any x).

Alternatively. We can also plug in values for x to get equations in A, B, C, D . Unfortunately, our only "magic" choice is $x = 0$. This gives $B = 1$. Instead of plugging in random values for x (we could do that!) we can then subtract the $(x^2+1)B$ from both sides and divide by x to get the simpler $x^2 - x + 3 = (x^2+1)A + x(Cx+D)$. Then we can again set $x = 0$ to find $3 = A$. Finish this for practice!

- Consequently: $\frac{x^7+3x+1}{x^4+x^2} = x^3 - x + \frac{3}{x} + \frac{1}{x^2} + \frac{-2x-1}{x^2+1}$.

Finally, we can integrate to find:

$$\begin{aligned}\int \frac{x^7 + 3x + 1}{x^4 + x^2} dx &= \int \left(x^3 - x + \frac{3}{x} + \frac{1}{x^2} - \frac{2x}{x^2 + 1} - \frac{1}{x^2 + 1} \right) dx \\ &= \frac{1}{4}x^4 - \frac{1}{2}x^2 + 3\ln|x| - \frac{1}{x} - \ln(x^2 + 1) - \arctan(x) + C\end{aligned}$$

Here, we computed $\int \frac{2x}{x^2 + 1} dx = \ln(x^2 + 1) + C$ by substituting $u = x^2 + 1$. Do it!