

Review. Partial fractions reverses the process of putting fractions on a common denominator like in

$$\frac{5}{x-2} - \frac{4}{x+3} = \frac{5(x+3) - 4(x-2)}{(x-2)(x+3)} = \frac{x+23}{(x-2)(x+3)}.$$

Note that the numerator $x+23$ has degree 1 which is lower than the degree of the denominator (which is 2). This will always be the case if the numerators on the left-hand sides are just constants. The first step of the recipe below therefore is to make sure that the numerator of our fraction is of degree lower than the denominator.

If it is not, then we do long division. Afterwards, the numerator degree will be strictly lower than the denominator degree.

Example 108. Evaluate $\int \frac{2x-5}{x(x+1)(x+2)} dx$ by partial fractions.

Solution. In this example, we don't need to do long division first because the numerator is of degree lower than the denominator.

Partial fractions tells us that $\frac{2x-5}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$ for certain numbers A, B, C .

- We multiply both sides with $x(x+1)(x+2)$ to clear denominators:

$$2x-5 = (x+1)(x+2)A + x(x+2)B + x(x+1)C$$

- Set $x=0$ to get $-5=2A$ so that $A=-\frac{5}{2}$.
Set $x=-1$ to get $-7=-B$ so that $B=7$.
Set $x=-2$ to get $-9=2C$ so that $C=-\frac{9}{2}$.

- Consequently: $\frac{2x-5}{x(x+1)(x+2)} = -\frac{5}{2} \cdot \frac{1}{x} + \frac{7}{x+1} - \frac{9}{2} \cdot \frac{1}{x+2}$.

Hence: $\int \frac{2x-5}{x(x+1)(x+2)} dx = -\frac{5}{2} \int \frac{dx}{x} + 7 \int \frac{dx}{x+1} - \frac{9}{2} \int \frac{dx}{x+2} = -\frac{5}{2} \ln|x| + 7 \ln|x+1| - \frac{9}{2} \ln|x+2| + C.$

(recipe) To decompose the rational function $\frac{f(x)}{g(x)}$ into partial fractions:

- Check that $\text{degree } f(x) < \text{degree } g(x)$. (Otherwise, long division!)
- Factor $g(x)$ as far as possible.
- For each factor of $g(x)$ collect terms as follows:
 - For a linear factor $x-r$, occurring as $(x-r)^m$ in $g(x)$, these terms are

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \dots + \frac{A_m}{(x-r)^m}.$$
 - For a quadratic factor x^2+px+q , occurring as $(x^2+px+q)^m$ in $g(x)$, these terms are

$$\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \dots + \frac{B_mx+C_m}{(x^2+px+q)^m}.$$
- Determine the values of the unknown constants (the A 's, B 's and C 's).

Example 109. Evaluate $\int \frac{x^4 + 3x^3 + 1}{x^3 - x} dx$.

Solution.

- Since the degree of the numerator is not less than the degree of the denominator, we first perform long division. In this case, we get

$$\frac{x^4 + 3x^3 + 1}{x^3 - x} = x + 3 + \frac{x^2 + 3x + 1}{x^3 - x}.$$

- Factor the denominator: $x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$.
- $\frac{x^4 + 3x^3 + 1}{x^3 - x} = x + 3 + \frac{x^2 + 3x + 1}{x(x - 1)(x + 1)}$
- By partial fractions $\frac{x^2 + 3x + 1}{x(x - 1)(x + 1)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}$ for certain numbers A, B, C .

- We multiply both sides with $x(x - 1)(x + 1)$ to clear denominators:

$$x^2 + 3x + 1 = (x - 1)(x + 1)A + x(x + 1)B + x(x - 1)C$$

- Set $x = 0$ to get $1 = -A$ so that $A = -1$.
Set $x = 1$ to get $5 = 2B$ so that $B = \frac{5}{2}$.
Set $x = -1$ to get $-1 = 2C$ so that $C = -\frac{1}{2}$.

Therefore,

$$\begin{aligned} \int \frac{x^4 + 3x^3 + 1}{x^3 - x} dx &= \int \left(x + 3 - \frac{1}{x} + \frac{5/2}{x - 1} - \frac{1/2}{x + 1} \right) dx \\ &= \frac{1}{2}x^2 + 3x - \ln|x| + \frac{5}{2}\ln|x - 1| - \frac{1}{2}\ln|x + 1|. \end{aligned}$$