

Example 104. Determine $\int \frac{1}{(1+x^2)^2} dx$. [That's an integral we care about for partial fractions!]

Solution. We substitute $x = \tan\theta$ because then $1+x^2 = \sec^2\theta$. Since $\frac{dx}{d\theta} = \frac{d}{d\theta}\tan\theta = \sec^2\theta$, we get

$$\int \frac{1}{(1+x^2)^2} dx = \int \frac{\sec^2\theta d\theta}{(\sec^2\theta)^2} = \int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta d\theta.$$

From Example 93, we know that

$$\int \cos^2\theta d\theta = \frac{1}{2}(\cos(\theta)\sin(\theta) + \theta) + C.$$

After replacing $\theta = \arctan(x)$, we could stop here, except that our answer can be considerably simplified!

For this, draw a right-angled triangle with angle θ . To encode the relationship $\tan\theta = \frac{\text{opp}}{\text{adj}} = x$, we assign the opposite side length x and the adjacent side length 1 as in the diagram to the right.

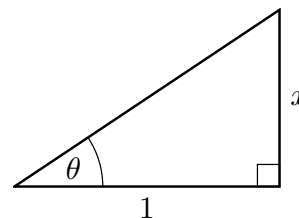
By Pythagoras, the hypotenuse then has length $\sqrt{1+x^2}$. It follows that

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+x^2}}, \quad \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{1+x^2}}.$$

Hence $\cos(\theta)\sin(\theta) = \frac{x}{1+x^2}$ so that, combined, we get

$$\int \frac{1}{(1+x^2)^2} dx = \frac{1}{2}(\cos(\theta)\sin(\theta) + \theta) + C = \frac{1}{2}\left[\frac{x}{1+x^2} + \arctan(x)\right] + C.$$

Comment. We just showed that, for instance, $\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$.



Partial fractions

Review. rational function = $\frac{\text{polynomial}}{\text{another polynomial}}$

Example 105. We are surely all familiar with putting stuff on a common demoninator like in

$$\frac{2}{x+1} + \frac{3}{x-1} = \frac{2(x-1) + 3(x+1)}{(x+1)(x-1)} = \frac{5x+1}{(x+1)(x-1)}.$$

Partial fractions refers to reversing this process of putting things on a common denominator.

Example 106. The previous example allows us to easily compute the following integral:

$$\int \frac{5x+1}{(x+1)(x-1)} dx = \int \frac{2}{x+1} dx + \int \frac{3}{x-1} dx = 2\ln|x+1| + 3\ln|x-1| + C.$$

Make sure that you are comfortable with integrating the two simpler integrals!

[For instance, notice that substituting $u = x + 1$ in the first of the two, we have $du = dx$ which is why we just get $\log$ of $x + 1$.]

Example 107. Evaluate $\int \frac{x+4}{x(x-2)} dx$ by partial fractions.

Solution. Partial fractions tells us that $\frac{x+4}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$ for some numbers A, B that we still need to find:

- To find A and B we multiply both sides with $x(x-2)$ to clear denominators:

$$x+4 = (x-2)A + xB$$

- This equation has to be true for all values of x :
Set $x = 0$ to get $4 = -2A$ so that $A = -2$.
Set $x = 2$ to get $6 = 2B$ so that $B = 3$.
- We can now verify that, indeed, $\frac{x+4}{x(x-2)} = \frac{-2}{x} + \frac{3}{x-2}$.

We therefore have $\int \frac{x+4}{x(x-2)} dx = \int \frac{-2}{x} dx + \int \frac{3}{x-2} dx = -2\ln|x| + 3\ln|x-2| + C.$

Important. Setting $x = 0$ and $x = 2$ makes our life particularly easy. On the other hand, note that we can set x to any values to get valid equations for A and B (once we have two equations for those two unknowns, we should be able to solve for them).

Alternatively, note that both sides of $x+4 = (x-2)A + xB$ are polynomials in x . We can therefore also equate coefficients: comparing the coefficients of x gives $1 = A + B$ while comparing constant coefficients gives $4 = -2A$. Solving these, we again find $A = -2$ and $B = 3$.