Example 31. Consider the region enclosed by the curves $y=\sqrt{x}$, $y=0$, $x=1$, $x=4$. If we revolve this region about the x-axis, what is the volume of the resulting solid?

Solution. Make a sketch! The solid should look roughly like a solid coffee mug (no room for coffee) without handles.

The cross-section at x is a disk with radius $R(x)=\sqrt{x}$ and so has area $\pi R(x)^2=\pi x$. If we give this disk a thickness of $\mathrm{d}x$, then its volume is $\pi R(x)^2\,\mathrm{d}x=\pi x\mathrm{d}x$. The total volume is

$$\text{vol}=\int_1^4 \pi R(x)^2\,\mathrm{d}x=\int_1^4 \pi x\,\mathrm{d}x=\pi\left[\frac{1}{2}x^2\right]_1^4=\pi\left(8-\frac{1}{2}\right)=\frac{15}{2}\pi.$$

Example 32. Consider the region enclosed by the curves $y=\sqrt{x}$, $y=1$, $x=1$, $x=4$. If we revolve this region about the line $y=1$, what is the volume of the resulting solid?

Solution. Again, make a sketch! This solid should look roughly like a bullet.

The cross-section at x now is a disk with radius $R(x)=\sqrt{x}-1$ and so has area $\pi R(x)^2=\pi(\sqrt{x}-1)^2$. If we give this disk a thickness of $\mathrm{d}x$, then its volume is $\pi R(x)^2\,\mathrm{d}x=\pi x\mathrm{d}x$. The total volume is

$$\text{vol}=\int_1^4 \pi R(x)^2\,\mathrm{d}x=\int_1^4 \pi\,(\sqrt{x}-1)^2\,\mathrm{d}x=\int_1^4 \pi\,(x-2\sqrt{x}+1)\,\mathrm{d}x=\pi\left[\frac{1}{2}x^2-\frac{4}{3}x^{3/2}+x\right]_1^4=\pi\left(\frac{4}{3}-\frac{1}{6}\right)=\frac{7}{6}\pi.$$

Example 33. Consider (again) the region enclosed by the curves $y=\sqrt{x}$, $y=1$, $x=1$, $x=4$. If we revolve this region about the x-axis, what is the volume of the resulting solid?

The solid should look roughly like the coffee mug from Example 31 with a cylindrical hole drilled out.

What do the cross-sections look like now?

Your final answer should be $\frac{9}{2}\pi$.

Solution. The solid should look roughly like the coffee mug from Example 31 with a cylindrical hole drilled out.

The cross-section at x now is a **washer** with outer radius $R(x)=\sqrt{x}$ and inner radius $r(x)=1$ so that its area is $\pi R(x)^2-\pi r(x)^2=\pi x-\pi=\pi(x-1)$. If we give this washer a thickness of $\mathrm{d}x$, then its volume is $\pi(x-1)\mathrm{d}x$. The total volume is

$$\text{vol}=\int_1^4 \pi\,(x-1)\,\mathrm{d}x=\pi\left[\frac{1}{2}x^2-x\right]_1^4=\pi\left(4-\left(-\frac{1}{2}\right)\right)=\frac{9}{2}\pi.$$

Comment. In this simple case, we could also obtain the final answer from Example 31 by simply subtracting the volume of the cylinder (base area $\pi\cdot 1^2$ times height $4-1=3$ gives a volume of 3π) that is "drilled out".

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Volumes using cylindrical shells

See Section 6.2 in our book for nice illustrations of cylindrical shells!

Example 34. Consider the region bounded by the curves $y = x^2$, $y = 0$ and $x = 2$. Determine the volume of the solid generated by revolving this region about the y-axis

(a) using horizontal cross-sections of the region (turning into washers), and

(b) using vertical cross-sections of the region (turning into **cylindrical shells**).

Solution. First, make a sketch as we did in class!

(a) Our region extends from $y = 0$ to $y = 4$. The cross-section at y (a line which extends from $x = \sqrt{y}$ to $x = 2$) turns into a washer with outer radius $R(y) = 2$ and inner radius $r(y) = \sqrt{y}$, which has area $\pi R(y)^2 - \pi r(y)^2 = \pi(4 - y)$. If we give this disk a thickness of $\mathrm{d}y$, then its volume is $\pi(4 - y)\mathrm{d}y$. The total volume is

$$\mathrm{vol} = \int_0^4 \pi(4 - y)\,\mathrm{d}y = \pi\left[4y - \frac{1}{2}y^2\right]_0^4 = 8\pi.$$

(b) Our region extends from $x = 0$ to $x = 2$. The cross-section at x (a line which extends from $y = 0$ to $y = x^2$) turns into a cylindrical shell with radius $r(x) = x$ and height $h(x) = x^2$ and so has area $2\pi r(x) \cdot h(x) = 2\pi x^3$. If we give this shell a thickness of $\mathrm{d}x$, then its volume is $2\pi x^3 \mathrm{d}x$. The total volume is

$$\mathrm{vol} = \int_0^2 2\pi x^3\,\mathrm{d}x = 2\pi\left[\frac{1}{4}x^4\right]_0^2 = 8\pi.$$

Comment. In this example, both horizontal and vertical cross-sections worked (because $y = x^2$ can be easily solved for $x = \sqrt{y}$ when $x \geqslant 0$). On the other hand, think about what happens when we replace $y = x^2$ with $y = f(x)$ where $f(x)$ is just some positive function. In that case, horizontal cross-sections would require us to work with the inverse function of $f(x)$ (which may not exist as such if $f(x)$ takes some y-values multiple times). On the other hand, vertical cross-sections would always work exactly as above and we would get

$$\mathrm{vol} = \int_0^2 2\pi x f(x)\,\mathrm{d}x$$

for the total volume.

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