Areas enclosed by curves

Theorem 25. The area enclosed by the curves $y = f(x)$ and $y = g(x)$, between $x = a$ and $x = b$, is given by

$$\int_a^b [f(x) - g(x)]\,\mathrm{d}x$$

provided that $f(x) \geqslant g(x)$ (for all $x \in [a, b]$).

[Note that the area is always $\int_a^b |f(x) - g(x)|\,\mathrm{d}x$ but to work with the absolute value, we need to break the problem into subcases according to whether $f(x) - g(x) \geqslant 0$ or $f(x) - g(x) \leqslant 0$.]

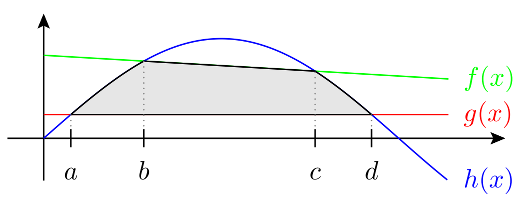
Example 26. What is the area enclosed by the curves $y = \cos(x)$, $y = 1$, $x = 0$, $x = 2\pi$?

First, write down an integral and compute its value, then look at your sketch (always make a quick sketch!) and confirm that your answer makes rough sense.

Solution. Note that $y = 1$ describes a horizontal line while $x = 0$ and $x = 2\pi$ are two vertical lines. For our area, $y = 1$ lies above $y = \cos(x)$. Therefore, the area in question is

$$\int_0^{2\pi} (1 - \cos(x))\mathrm{d}x = \Big[x - \sin(x)\Big]_0^{2\pi} = (2\pi - \sin(2\pi)) - (0 - \sin(0)) = 2\pi.$$

Example 27. Consider the plot below. What is the area enclosed by the curves $y = f(x)$, $y = g(x)$ and $y = h(x)$?



Solution. We split up the area into three smaller regions in which we can apply Theorem 25. The area is

$$\int_a^b [h(x) - g(x)]\mathrm{d}x + \int_b^c [f(x) - g(x)]\mathrm{d}x + \int_c^d [h(x) - g(x)]\mathrm{d}x.$$

Comment. In this case, we can alternatively (and equivalently) write the area as the difference

$$\int_a^d [h(x) - g(x)]\mathrm{d}x - \int_b^c [h(x) - f(x)]\mathrm{d}x.$$

Example 28. What is the area enclosed by the curves $y=2-x^2$, $y=-x$?

(a) First, make a sketch!

(b) Find intersections of the curves.

(c) Write down the integral for the area of interest.

(d) Evaluate the integral.

Solution.

(a) Do it! This should always be our first step.

(b) Since the equations for both curves are of the form $y=\ldots$, we can find the (x coordinates of the) intersections by setting the right-hand sides of the equations equal:

$$2-x^2=-x \quad\Longrightarrow\quad x^2-x-2=0 \quad\Longrightarrow\quad x=-1,2$$

For the final step, we solved the quadratic equation (for instance, using the quadratic formula).
[It's not needed for the remaining parts, but we can get the corresponding y-coordinates from either $y=2-x^2$ or $y=-x$. The latter is simpler and we find that the two intersections are $(-1,1)$ and $(2,-2)$.]

(c) This tells us (look at sketch!) that our area extends from $x=-1$ to $x=2$ and that the curve $y=2-x^2$ is the upper boundary while $y=-x$ is the lower boundary. Therefore, the area is

$$\int_{-1}^{2}((2-x^2)-(-x))\mathrm{d}x.$$

(d) We compute the area as

$$\int_{-1}^{2}((2-x^2)-(-x))\mathrm{d}x=\int_{-1}^{2}(2+x-x^2)\mathrm{d}x=\left[2x+\frac{1}{2}x^2-\frac{1}{3}x^3\right]_{-1}^{2}=\ldots=\frac{9}{2}.$$

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