Quiz. There will be a first quiz in lab on Thursday. One of the two problems will be to compute an integral like the following by substitution:

$$\int \cos^5(3t)\sin(3t)\mathrm{d}t$$

Homework. Compute this indefinite integral by substituting $u = \cos(3t)$.

Example 17. Determine $\displaystyle\int x^3\sqrt{x^2+1}\mathrm{d}x$.

Solution. We substitute $u = x^2+1$. Since $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, we have $x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.
Note that there will be x^2 left into the integral which we replace with $x^2 = u-1$.
The integral therefore becomes:

$$\begin{aligned}
\int x^3\sqrt{x^2+1}\mathrm{d}x &= \int \frac{1}{2}x^2\sqrt{u}\,\mathrm{d}u = \int \frac{1}{2}(u-1)\sqrt{u}\,\mathrm{d} \\
&= \frac{1}{2}\int (u^{3/2}-u^{1/2})\,\mathrm{d}u = \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + C \\
&= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C
\end{aligned}$$

Comment. If you prefer, the final answer could be rewritten as $\dfrac{1}{15}(x^2+1)^{3/2}(3x^2-2)+C$.

Example 18. Determine $\displaystyle\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t$.

Here, a very natural substitution is $u = 2-\cos(t)$.

(Note how we can already anticipate that the derivative will nicely take care of the $\sin(t)$ in the integrand.)

Solution. We again have the choice of either substituting without limits or with limits:

(a) We substitute $u = 2-\cos(t)$. Since $\frac{\mathrm{d}u}{\mathrm{d}t} = \sin(t)$, we get $\sin(t)\mathrm{d}t = \mathrm{d}u$. Hence,

$$\int \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \int \frac{1}{u}\mathrm{d}u = \ln|u| + C = \ln|2-\cos(t)| + C.$$

Hence (by the Fundamental Theorem of Calculus)

$$\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \Big[\ln|2-\cos(t)|\Big]_0^\pi = \ln(3).$$

(b) Alternatively, once we feel comfortable with integration, we can do this substitution in a single step by also adjusting the boundaries. When $t=0$ we have $u = 2-\cos(0) = 1$, and when $t=\pi$ we have $u = 2-\cos(\pi) = 3$. Therefore,

$$\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \int_1^3 \frac{1}{u}\mathrm{d}u = \Big[\ln|u|\Big]_1^3 = \ln(3).$$

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Extra practice

Example 19. $\displaystyle\int \frac{\mathrm{d}x}{x\ln(x)} =$

First, use the substitution $u = \ln(x)$. (Can you already see why this is a good choice?)

Then, for practice, use $u = \frac{1}{\ln(x)}$ and see if you can get the same final answer.

(For more complicated integrals, finding the "best" substitution is quite an art form. For the integrals we are concerned with, there is always a natural choice.)

Example 20. $\displaystyle\int_2^4 \frac{\mathrm{d}x}{x\ln(x)} =$

To reduce work, use the previous problem and the fundamental theorem.

Example 21. $\displaystyle\int x\sin(x^2+3)\,\mathrm{d}x =$

Example 22. $\displaystyle\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)\mathrm{d}x =$

Example 23. $\displaystyle\int 3x^5\sqrt{x^3+1}\,\mathrm{d}x =$

First, use the substitution $u = x^3+1$ (that's the most natural choice).

Then, for practice, use $u = \sqrt{x^3+1}$ and see if you can get the same final answer.

Example 24.

(a) Determine $\displaystyle\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t$.

(b) Determine $\displaystyle\int_0^\pi \frac{\sin^3(t)}{2-\cos(t)}\mathrm{d}t$.

Solution.

(a) **(again)** We substitute $u = 2-\cos(t)$. When $t=0$ we have $u = 2-\cos(0) = 1$, and when $t=\pi$ we have $u = 2-\cos(\pi) = 3$. Therefore,

$$\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \int_1^3 \frac{1}{u}\mathrm{d}u = \Big[\ln|u|\Big]_1^3 = \ln(3).$$

(b) We substitute $u = 2-\cos(t)$ again. When $t=0$ we have $u = 2-\cos(0) = 1$, and when $t=\pi$ we have $u = 2-\cos(\pi) = 3$. This time, there will be $\sin^2(t)$ left over in the integral so we need to rewrite this in terms of u.

Note that $\cos(t) = 2-u$. (At this point, we could solve for t to get $t = \arccos(2-u)$ and use this to substitute away any remaining t. However, we would get $\sin^2(t) = \sin^2(\arccos(2-u))$ which is not pleasant and would have to be simplified. We get this simplification for free by proceeding slightly differently.) Recall that $\cos^2(t)+\sin^2(t) = 1$ so that $\sin^2(t) = 1-\cos^2(t) = 1-(2-u)^2 = -u^2+4u-3$.

Therefore,

$$\int_0^\pi \frac{\sin^3(t)}{2-\cos(t)}\mathrm{d}t = \int_1^3 \frac{\sin^2(t)}{u}\mathrm{d}u = \int_1^3 \frac{-u^2+4u-3}{u}\mathrm{d}u = \int_1^3 \left(-u+4-\frac{3}{u}\right)\mathrm{d}u = \ldots = 4-3\ln(3).$$

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