Quiz #3

MATH 126 — Calculus II
Thursday, Sept 12

Please print your name:

Problem 1. Consider the region in the first quadrant bounded by the curves $y = x^2$, $y = 2 - x^2$ and $x = 0$.

Use the shell method to find the volume of the solid generated by revolving this region about the y-axis.

Solution. First, make a sketch! Note that the region extends from $x = 0$ to $x = 1$ (if $x = 1$ is not clear from the sketch, you can compute the intersection of $y = x^2$, $y = 2 - x^2$: $x^2 = 2 - x^2$ simplifies to $x^2 = 1$ or $x = \pm 1$).

The shell at position x has radius x and height $(2 - x^2) - x^2 = 2 - 2x^2$. Given a thickness of $\mathrm{d}x$, it therefore has (approximate) volume $2\pi x(2 - 2x^2)\mathrm{d}x$. Hence, the total volume is

$$\int_0^1 2\pi x(2 - 2x^2)\mathrm{d}x = 4\pi \int_0^1 (x - x^3)\mathrm{d}x = 4\pi \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = 4\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \pi.$$

Alternatively. Here is how the disk/washer method looks like for the same problem. In this case, we consider horizontal cross-sections from $y = 0$ to $y = 2$ (see sketch). If $0 \leqslant y \leqslant 1$, then we have disks of radius $\sqrt{y}$ ($y = x^2$ is equivalent to $x = \sqrt{y}$). On the other hand, if $1 \leqslant y \leqslant 2$, then we have disks of radius $\sqrt{2 - y}$ ($y = 2 - x^2$ is equivalent to $x = \sqrt{2 - y}$). Overall, the total volume is again

$$\int_0^1 \pi(\sqrt{y})^2\mathrm{d}y + \int_1^2 \pi(\sqrt{2 - y})^2\mathrm{d}y = \pi\int_0^1 y\mathrm{d}y + \pi\int_1^2 (2 - y)\mathrm{d}y = \pi\left[\frac{1}{2}y^2\right]_0^1 + \pi\left[2y - \frac{1}{2}y^2\right]_1^2 = \frac{1}{2}\pi + \pi\left(2 - \frac{3}{2}\right) = \pi.$$

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