Midterm #2

MATH 126 — Calculus II
Thursday, Oct 24, 2024

Please print your name:

No notes, graphing calculators or other tools are permitted. There are 30 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (6 points) Evaluate the following indefinite integral: $\int \cos^6(2x)\sin^3(2x)\,\mathrm{d}x$

Solution. We substitute $u=\cos(2x)$, because then $\frac{\mathrm{d}u}{\mathrm{d}x}=-2\sin(2x)$ and thus $\sin(2x)\mathrm{d}x=-\frac{1}{2}\mathrm{d}u$, to get

$$\begin{aligned}\int \cos^6(2x)\sin^3(2x)\,\mathrm{d}x &= -\frac{1}{2}\int u^6\sin^2(2x)\,\mathrm{d}u = -\frac{1}{2}\int u^6(1-\cos^2(2x))\,\mathrm{d}u = -\frac{1}{2}\int u^6(1-u^2)\mathrm{d}u\\ &= -\frac{1}{2}\left(\frac{u^7}{7}-\frac{u^9}{9}\right)+C = \frac{\cos^9(2x)}{18}-\frac{\cos^7(2x)}{14}+C.\end{aligned}$$

Problem 2. (4 points) Determine the shape (but not the exact numbers involved) of the partial fraction decompositions of:

(a) $\dfrac{2x-6}{x(x+1)(x+2)}=$ ☐

(b) $\dfrac{x^6}{(x+2)^2(x^2+1)}=$ ☐

Solution.

(a) $\dfrac{2x-6}{x(x+1)(x+2)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x+2}$

(b) Since the numerator degree is 6 but the denominator degree is 4, we have to do long division first. This will result in Ax^2+Bx+C plus a remainder. Overall:

$$\frac{x^6}{(x+2)^2(x^2+1)}=Ax^2+Bx+C+\frac{D}{x+2}+\frac{E}{(x+2)^2}+\frac{Fx+G}{x^2+1}$$

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Problem 3. (6 points) Evaluate the integral $\int_1^2 x\ln(x)\,\mathrm{d}x$.

Solution. We apply integration by parts, and use $\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$ with $f(x) = \ln(x)$ and $g'(x) = x$. With $g(x) = \frac{1}{2}x^2$, we then get

$$\int_1^2 x\ln(x)\,\mathrm{d}x = \left[\frac{1}{2}x^2\ln(x)\right]_1^2 - \int_1^2 \frac{1}{x}\cdot\frac{1}{2}x^2\,\mathrm{d}x = 2\ln(2) - \frac{1}{2}\int_1^2 x\,\mathrm{d}x = 2\ln(2) - \frac{1}{2}\left[\frac{1}{2}x^2\right]_1^2 = 2\ln(2) - \frac{3}{4}.$$

Problem 4. (4 points) Evaluate the following indefinite integral: $\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x$.

Solution. We substitute $x = 3\sin\theta$ because then $9 - x^2 = 9(1-\sin^2\theta) = 9\cos^2\theta$. Since $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\cos\theta$, we get

$$\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{9\cos^2\theta}}3\cos\theta\mathrm{d}\theta = \int 1\,\mathrm{d}\theta = \theta + C = \arcsin\left(\frac{x}{3}\right) + C.$$

Problem 5. (7 points) Evaluate the following indefinite integral: $\int \frac{2x-3}{x^2+3x} \mathrm{d}x$

Solution. Partial fractions tells us that $\frac{2x-3}{x^2+3x} = \frac{2x-3}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$ for some numbers A, B that we still need to find:

- To find A and B we multiply both sides with $x(x+3)$ to clear denominators:

$$2x-3 = (x+3)A + xB$$

- Set $x=0$ to get $-3=3A$ so that $A=-1$.

  Set $x=-3$ to get $-9=-3B$ so that $B=3$.

We therefore have $\int \frac{2x-3}{x^2+3x} \mathrm{d}x = \int \left(\frac{-1}{x} + \frac{3}{x+3}\right) \mathrm{d}x = -\ln|x| + 3\ln|x+3| + C$.

Problem 6. (3 points) Evaluate the following integral or show that it diverges: $\int_{-1}^{1} \frac{1}{2x+1} \mathrm{d}x$

Solution. Note that the integrand has a singularity at $x = -\frac{1}{2}$. The antiderivative is

$$\int \frac{1}{2x+1} \mathrm{d}x = \frac{1}{2}\ln|2x+1| + C,$$

and we can already see that we will not get a finite contribution for $x = -\frac{1}{2}$ since $\lim_{x \to -1/2} \frac{1}{2}\ln|2x+1| = -\infty$. This means that our integral diverges.

(extra scratch paper)

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