Midterm #2 – Practice

MATH 126 — Calculus II
Midterm: Thursday, Oct 24, 2024

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any mathematical typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Reminder. A nongraphing calculator (equivalent to the TI-30XIIS) is allowed on the exam (but not needed). No notes or further tools of any kind will be permitted on the midterm exam.

Problem 1. Go over all the quizzes since the first midterm exam!

To help you with that, there is a version of each quiz posted on our course website without solutions (of course, there are solutions, too).

Problem 2. Evaluate the integral $\int_1^2 x^3 \ln(x)\,\mathrm{d}x$.

Solution. We apply integration by parts, and use $\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$ with $f(x) = \ln(x)$ and $g'(x) = x^3$. With $g(x) = \frac{1}{4}x^4$, we then get

$$\begin{aligned}\int_1^2 x^3 \ln(x)\,\mathrm{d}x &= \left[\frac{1}{4}x^4 \ln(x)\right]_1^2 - \int_1^2 \frac{1}{x}\cdot\frac{1}{4}x^4\,\mathrm{d}x = 4\ln(2) - \frac{1}{4}\int_1^2 x^3\,\mathrm{d}x \\ &= 4\ln(2) - \frac{1}{4}\left[\frac{1}{4}x^4\right]_1^2 = 4\ln(2) - \frac{1}{4}\left(4 - \frac{1}{4}\right) = 4\ln(2) - \frac{15}{16}.\end{aligned}$$

Problem 3. Determine the shape (but not the exact numbers involved) of the partial fraction decompositions of:

(a) $\dfrac{3x^2 - 2}{x^2(x-1)(x+2)}$

(b) $\dfrac{x^9 + 1}{x(x+2)(x^2+1)^2}$

Solution.

(a) $\dfrac{3x^2 - 2}{x^2(x-1)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{x+2}$

(b) Since the numerator degree is 9 but the denominator degree is 6, we have to do long division first. This will result in $Ax^3 + Bx^2 + Cx + D$ plus a remainder. Overall:

$$\frac{x^9 + 1}{x(x+2)(x^2+1)^2} = Ax^3 + Bx^2 + Cx + D + \frac{E}{x} + \frac{F}{x+2} + \frac{Gx + H}{x^2+1} + \frac{Ix + J}{(x^2+1)^2}$$

Problem 4. Evaluate $\int_3^4 \frac{x^3 - 2x^2 - 4}{x^3 - 2x^2}\,\mathrm{d}x$.

Solution. First, we do long division to obtain

$$\frac{x^3 - 2x^2 - 4}{x^3 - 2x^2} = 1 - \frac{4}{x^3 - 2x^2}.$$

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Next, we factor the denominator:

$$x^3 - 2x^2 = x^2(x-2).$$

We know that we can find numbers A, B, C such that

$$\frac{4}{x^3 - 2x^2} = \frac{4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}.$$

Clearing denominators, we get

$$4 = x(x-2)A + (x-2)B + x^2 C.$$

Setting $x = 2$, we find $4 = 4C$, so $C = 1$. Setting $x = 0$, we find $4 = -2B$, so $B = -2$. There's no great third choice for x, so we plug in any value, say, $x = 1$ to find $4 = -A - B + C = -A + 2 + 1$. Hence, $A = -1$.

Therefore,

$$\begin{aligned}\int_3^4 \frac{x^3 - 2x^2 - 4}{x^3 - 2x^2}\,\mathrm{d}x &= \int_3^4 1 + \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x-2}\,\mathrm{d}x \\ &= \left[x + \ln|x| - \frac{2}{x} - \ln|x-2|\right]_3^4 = \left(4 + \ln 4 - \frac{1}{2} - \ln 2\right) - \left(3 + \ln 3 - \frac{2}{3} - \ln 1\right) \\ &= \frac{7}{6} + \ln\frac{2}{3}.\end{aligned}$$

(In the final simplification, we used that $\ln 4 = \ln(2^2) = 2\ln 2$ as well as $\ln 2 - \ln 3 = \ln\frac{2}{3}$.)

Problem 5. Evaluate the following indefinite integrals:

(a) $\displaystyle\int \cos^4(5x)\sin^3(5x)\,\mathrm{d}x$

(b) $\displaystyle\int \cos^5(3x)\sin^2(3x)\,\mathrm{d}x$

Solution.

(a) We substitute $u = \cos(5x)$, because then $\frac{\mathrm{d}u}{\mathrm{d}x} = -5\sin(5x)$ and thus $\sin(5x)\mathrm{d}x = -\frac{1}{5}\mathrm{d}u$, to get

$$\begin{aligned}\int \cos^4(5x)\sin^3(5x)\,\mathrm{d}x &= -\frac{1}{5}\int u^4\sin^2(5x)\,\mathrm{d}u = -\frac{1}{5}\int u^4(1-\cos^2(5x))\,\mathrm{d}u = -\frac{1}{5}\int u^4(1-u^2)\mathrm{d}u \\ &= -\frac{1}{5}\left(\frac{u^5}{5} - \frac{u^7}{7}\right) + C = \frac{\cos^7(5x)}{35} - \frac{\cos^5(5x)}{25} + C.\end{aligned}$$

(b) We substitute $u = \sin(3x)$, because then $\frac{\mathrm{d}u}{\mathrm{d}x} = 3\cos(3x)$ and thus $\cos(3x)\mathrm{d}x = \frac{1}{3}\mathrm{d}u$, to get

$$\begin{aligned}\int \cos^5(3x)\sin^2(3x)\,\mathrm{d}x &= \frac{1}{3}\int \cos^4(3x)u^2\,\mathrm{d}u = \frac{1}{3}\int (1-\sin^2(3x))^2u^2\,\mathrm{d}u = \frac{1}{3}\int (1-u^2)^2u^2\,\mathrm{d}u \\ &= \frac{1}{3}\int (1-2u^2+u^4)u^2\,\mathrm{d}u = \frac{1}{3}\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C \\ &= \frac{1}{3}\left(\frac{\sin^3(3x)}{3} - \frac{2\sin^5(3x)}{5} + \frac{\sin^7(3x)}{7}\right) + C.\end{aligned}$$

Problem 6. Evaluate the following indefinite integrals:

(a) $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{1}{x^2\sqrt{9x^2-4}}\,\mathrm{d}x$

Solution.

(a) We substitute $x = \frac{2}{3}\sin\theta$ because then $4-9x^2 = 4(1-\sin^2\theta) = 4\cos^2\theta$. Since $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{2}{3}\cos\theta$, we get

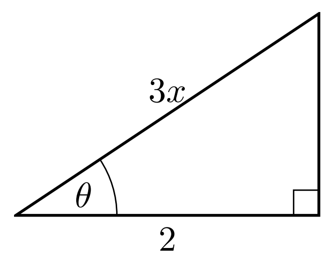
$$\int \frac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{4\cos^2\theta}}\,\frac{2}{3}\cos\theta\mathrm{d}\theta = \int \frac{1}{3}\,\mathrm{d}\theta = \frac{\theta}{3} + C = \frac{1}{3}\arcsin\left(\frac{3x}{2}\right) + C.$$

(b) This time we substitute $x = \frac{2}{3}\sec\theta$ because then $9x^2-4 = 4(\sec^2\theta - 1) = 4\tan^2\theta$. Since $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{2}{3}\sec\theta = \frac{2}{3}\sec\theta\tan\theta$ (you can work this out from $\sec\theta = \frac{1}{\cos\theta}$), we get

$$\int \frac{1}{x^2\sqrt{9x^2-4}}\,\mathrm{d}x = \int \frac{1}{\frac{4}{9}\sec^2\theta\sqrt{4\tan^2\theta}}\,\frac{2}{3}\sec\theta\tan\theta\mathrm{d}\theta = \frac{3}{4}\int \frac{1}{\sec\theta}\mathrm{d}\theta = \frac{3}{4}\int \cos\theta\mathrm{d}\theta = \frac{3}{4}\sin\theta + C.$$

Our final step consists in simplifying $\sin\theta$ given that $x = \frac{2}{3}\sec\theta$.

For this, draw a right-angled triangle with angle θ. To encode the relationship $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{3x}{2}$, we assign the hypothenuse length $3x$ and the adjacent side length 2 as in the diagram to the right.



By Pythagoras, the opposite side then has length $\sqrt{9x^2-4}$. It follows that

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{9x^2-4}}{3x}.$$

Overall, we have therefore found that

$$\int \frac{1}{x^2\sqrt{9x^2-4}}\,\mathrm{d}t = \frac{3}{4}\frac{\sqrt{9x^2-4}}{3x} + C = \frac{\sqrt{9x^2-4}}{4x} + C.$$

Problem 7. Evaluate the following integrals or show that they diverge.

(a) $\displaystyle\int_{-2}^{2} \frac{1}{x+1}\,\mathrm{d}x$

(b) $\displaystyle\int_{2}^{3} \frac{1}{\sqrt{3x-6}}\,\mathrm{d}x$

Solution.

(a) Note that the integrand has a singularity at $x=-1$. The antiderivative is

$$\int \frac{1}{x+1}\,\mathrm{d}x = \ln|x+1| + C,$$

and we can already see that we will not get a finite contribution for $x=-1$ since $\lim_{x\to -1}\ln|x+1| = -\infty$. This means that our integral diverges.

If we want to be more precise, we split the original integral into

$$\int_{-2}^{2} \frac{1}{x+1}\,\mathrm{d}x = \int_{-2}^{-1} \frac{1}{x+1}\,\mathrm{d}x + \int_{-1}^{2} \frac{1}{x+1}\,\mathrm{d}x$$

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so that neither integral has a singularity between its limits. Then, for instance, the first of the these integrals is

$$\int_{-2}^{-1} \frac{1}{x+1}\,\mathrm{d}x = \lim_{b\to -1^-} \int_{-2}^{b} \frac{1}{x+1}\,\mathrm{d}x = \lim_{b\to -1^-} \ln|x+1|\Big|_{-2}^{b} = \lim_{b\to -1^-} \ln|b+1| = -\infty,$$

and thus diverges. This means that the original integral $\int_{-2}^{2} \frac{1}{x+1}\,\mathrm{d}x$ diverges as well.

(b) Note that the integrand has a singularity at $x=2$. The antiderivative is

$$\int \frac{1}{\sqrt{3x-6}}\,\mathrm{d}x = \frac{2}{3}\sqrt{3x-6} + C,$$

and we can already see that we get a finite contribution for $x=2$. This means that our integral converges. It also means that we don't need to split the integral to compute its value (because the contributions for $x=2$ will just cancel out):

$$\int_{2}^{3} \frac{1}{\sqrt{3x-6}}\,\mathrm{d}x = \left[\frac{2}{3}\sqrt{3x-6}\right]_{2}^{3} = \frac{2}{3}\sqrt{3}$$

Problem 8. Evaluate $\int x^3 \cos(x^2+1)\,\mathrm{d}x$.

Solution. First, we substitute $t = x^2+1$ (so that $\mathrm{d}t = 2x\,\mathrm{d}x$) to get

$$\int x^3 \cos(x^2+1)\,\mathrm{d}x = \frac{1}{2}\int x^2 \cos(t)\mathrm{d}t = \frac{1}{2}\int (t-1)\cos(t)\mathrm{d}t.$$

Next, integrate by parts with $f(t) = t-1$ and $g'(t) = \cos(t)$. With $g(t) = \sin(t)$, we find

$$\int (t-1)\cos(t)\mathrm{d}t = (t-1)\sin(t) - \int \sin(t)\mathrm{d}t = (t-1)\sin(t) + \cos(t) + C.$$

Substituting back $t = x^2+1$, we finally obtain

$$\int x^3 \cos(x^2+1)\,\mathrm{d}x = \frac{1}{2}(x^2 \sin(x^2+1) + \cos(x^2+1)) + C.$$