Midterm #1

MATH 126 — Calculus II
Thursday, Sept 26, 2024

Please print your name:

No notes, graphing calculators or other tools are permitted. There are 30 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (3 points) Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{\mathrm{d}x}{2x} =$

(b) $\displaystyle\int \sin(5x)\,\mathrm{d}x =$

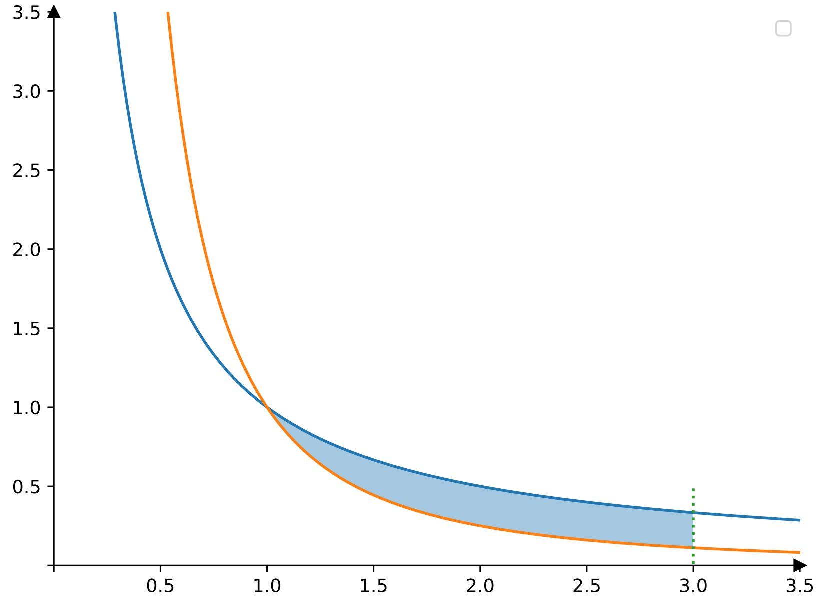
(c) $\displaystyle\int \frac{\mathrm{d}x}{x^2+1} =$

Solution. $\displaystyle\int \frac{\mathrm{d}x}{x} = 2\ln|x| + C$, $\displaystyle\int \sin(5x)\,\mathrm{d}x = -\frac{1}{5}\cos(5x) + C$, $\displaystyle\int \frac{\mathrm{d}x}{x^2+1} = \arctan(x) + C$

Problem 2. (5 points) Using the shell method, set up an integral (but do not evaluate it) for the volume of the solid obtained by revolving about the y-axis the region (in the first quadrant) enclosed by the curves

$$y = \frac{1}{x}, \quad y = \frac{1}{x^2}, \quad x = 3.$$

Solution. Here is a sketch:



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The two curves intersect at $x=1$ (this is visible from the sketch, or follows from equating $\frac{1}{x}=\frac{1}{x^2}$). The region therefore extends from $x=1$ to $x=3$. In that interval, $\frac{1}{x}$ is bigger than $\frac{1}{x^2}$.

The shell at position x has radius x and height $\frac{1}{x}-\frac{1}{x^2}$. Given a thickness of $\mathrm{d}x$, it therefore has (approximate) volume $2\pi x\left(\frac{1}{x}-\frac{1}{x^2}\right)\mathrm{d}x$. Hence, the total volume is

$$\int_1^3 2\pi x\left(\frac{1}{x}-\frac{1}{x^2}\right)\mathrm{d}x.$$

Of course, it would not be hard to continue and evaluate this integral.

Problem 3. (2 points) Set up an integral (but do not evaluate it) for the length of the curve $y=x^3$ for $1\leqslant x\leqslant 2$.

Solution.

$$\int_1^2 \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x=\int_1^2 \sqrt{1+(3x^2)^2}\,\mathrm{d}x=\int_1^2 \sqrt{1+9x^4}\,\mathrm{d}x$$

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Problem 4. (5 points) Evaluate the following indefinite integral: $\int \cos(3t)\sin^5(3t)\mathrm{d}t$

Solution. We substitute $u = \sin(3t)$. Since

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 3\cos(3t),$$

we use $\cos(3t)\,\mathrm{d}t = \frac{1}{3}\mathrm{d}u$ to get

$$\int \cos(3t)\sin^5(3t)\mathrm{d}t = \frac{1}{3}\int u^5\,\mathrm{d}u = \frac{1}{18}u^6 + C = \frac{1}{18}\sin^6(3t) + C.$$

Problem 5. (5 points) Evaluate the integral $\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,\mathrm{d}x$.

Solution. We substitute $u = x^3 + 1$, so that $\mathrm{d}u = 3x^2\mathrm{d}x$, and get

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,\mathrm{d}x = \frac{1}{3}\int_1^9 \frac{1}{\sqrt{u}}\,\mathrm{d}u = \left[\frac{2}{3}u^{1/2}\right]_1^9 = 2 - \frac{2}{3} = \frac{4}{3}.$$

Problem 6. (5 points) Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2, \;\; y(0) = 1.$

Solution. We separate variables,

$$\frac{1}{y^2}\,\mathrm{d}y = \mathrm{d}x$$

and integrate

$$\int \frac{1}{y^2}\,\mathrm{d}y = \int\,\mathrm{d}x$$

to find

$$-\frac{1}{y} = x + C.$$

Plugging in $y = 1$ and $x = 0$, we find $C = -1$. Solving for y, we find that

$$y(x) = -\frac{1}{x-1} = \frac{1}{1-x}.$$

Problem 7. (5 points) A conical container of radius 10 ft and height 20 ft is completely filled with water (the tip of the cone is at the bottom). Write down an integral for how much work it will take to pump the water to a level of 5 ft above the cone's rim. Do not evaluate the integral. [Water weighs 62.4 lb/ft³.]

Solution. Make a sketch! Let us denote with y (in ft) the vertical position in such a way that the tip is at $y = 0$ and the rim is at $y = 20$. We need to pump the water to the level $y = 20 + 5 = 25$.

We consider a horizontal slice at height y and thickness $\mathrm{d}y$:

- This slice needs to be lifted up $25 - y$ (ft).
- The slice is (almost) a disk with radius $r = \frac{10}{20}y = \frac{y}{2}$. Hence, its volume is $\pi r^2\,\mathrm{d}y = \frac{\pi}{4}y^2\mathrm{d}y$ (ft^3).
- The weight of the slice is $62.4\,\frac{\pi}{4}y^2\,\mathrm{d}y$ (lb) and it needs to be lifted up $25 - y$ (ft). That takes work of $(25 - y)\cdot 62.4\,\frac{\pi}{4}y^2\mathrm{d}y$ (ft-lb).
- "Adding" up, the total work required is

  $$\int_0^{20}(25-y)\cdot 62.4\,\frac{\pi}{4}y^2\mathrm{d}y,$$

  which is about $1.31\cdot 10^6$ ft-lb (but we were asked to not evaluate the integral).

Problem 8. (tiny bonus!) Very roughly, what is the distance from us to the moon?

Solution. Roughly, $384,000$ km or $239,000$ miles. Equivalently, since the speed of light is about $300,000$ km/s, this distance is about 1.28 light-seconds.

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(extra scratch paper)